Solutions to Problem Set 6

ECON 337901 - Financial Economics — Peter Ireland
Boston College, Department of Economics — Spring 2019

Due Tuesday, February 26

1. Stocks, Bonds, and Contingent Claims

Let s^G denote the number of shares of stock and b^G the number of bonds in the portfolio that replicates the payoffs provided by the contingent claim for the good state. Since the stock pays off 2 and the bond pays off 1 in the good state, we need

$$2s^G + b^G = 1$$

to replicate the contingent claim's payoff of 1 in the good state. And since the stock pays off 1 and the bond pays off one in the bad state, we need

$$s^G + b^G = 0$$

to replicate the contingent claim's payoff of 0 in the bad state. This is a system of two equations in two unknowns, which can be solved in a variety of ways. One way is just to subtract the second equation from the first to get

$$s^G = 1$$

and then to use either of the two individual equations to see that

$$b^G = -1.$$

Similarly, let s^B denote the number of shares of stock and b^B the number of bonds in the portfolio that replicates the payoffs provided by the contingent claim for the bad state. Since the stock pays off 2 and the bond pays off 1 in the good state, we need

$$2s^B + b^B = 0$$

to replicate the contingent claim's payoff of 0 in the good state. And since the stock pays off 1 and the bond pays off one in the bad state, we need

$$s^B + b^B = 1$$

to replicate the contingent claim's payoff of 1 in the bad state. Again, we have a system of two equations in two unknowns, which can be solved in a variety of ways. If, as before, we simply subtract the second equation from the first to get

$$s^B = -1,$$

we can then to use either of the two individual equations to see that

$$b^B = 2.$$

Now it only remains to find the prices of the two contingent claims. Since the payoffs made by the contingent claim for the good state can be replicated by the portfolio that sets $s^G = 1$ and $b^G = -1$, the claim must sell at $t = 0$ for the same price as that portfolio. Hence, the claim price is

$$q^G = q^s - q^b = 1.1 - 0.9 = 0.2.$$

Similarly, since the payoffs made by the contingent claim for the bad state can be replicated by the portfolio that sets $s^B = -1$ and $b^B = 2$, the claim price must be

$$q^B = 2q^b - q^s = 1.8 - 1.1 = 0.7.$$

2. Option Pricing

Robert Merton suggested that the option pricing problem can be solved by creating a portfolio consisting of s shares of stock and b bonds that replicates the payoff on the option with strike price K. We now know that the values of s and b must solve the two equations

$$2 - K = 2s + b$$

and

$$0 = s + b$$

required to replicate the option's payoff of $2 - K$ in the good state and zero in the bad state.

Although there are many ways to solve this system of two equations in two unknowns, one way it to subtract that second equation from the first, to get

$$s = 2 - K,$$

and then to substitute this solution back into the second equation to get

$$b = -s = K - 2.$$

Since K is assumed to lie between 1 and 2, this portfolio involves taking a long position in the stock and a short position in bonds. Equivalently, we can think of this strategy as borrowing in order to buy shares of stock, that is, buying the stock on margin.

We know that the stock sells for $q^s = 1.1$ and the bond for $q^b = 0.9$ at $t = 0$. Therefore, the cost of assembling this portfolio is

$$q^s s + q^b b = 1.1(2 - K) + 0.9(K - 2) = 2.2 - 1.1K + 0.9K - 1.8 = 0.4 - 0.2K$$

and, if there are no arbitrage opportunities across the stock, bond, and option markets, this will be the price of the option with strike price K as well. Note that the option price declines as K goes up. This makes sense: an option with a higher strike price is less attractive, as it requires the holder to pay a higher price for the stock of he or she chooses to exercise the option at $t = 1$. Note, as well, that if the strike price is $K = 2$, the implied option price is zero. This makes sense, too: in this case, the payoff from the option is zero in both states at $t = 1$, and no one would be willing to pay a positive price for the option at $t = 0$.

An even easier way to price the option is to observed that its payoffs can be replicated more simply by a portfolio that includes $2 - K$ contingent claims for the good state and zero contingent claims for the bad state. In answering question 1, we found that the price of a claim for the good state is $q^G = 0.2$. Therefore, the cost of $2 - K$ contingent claims for the good state is

$$q^G(2 - K) = 0.2(2 - K) = 0.4 - 0.2K,$$

which confirms that Merton's approach gives the same answer for the option price as one that is based on portfolios of contingent claims.