Solutions to Problem Set 13

ECON 337901 - Financial Economics
Boston College, Department of Economics

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For Extra Practice - Not Collected or Graded

1. Portfolio Allocation with Mean-Variance Utility, Part I

The first-order condition for choosing σ_P to maximize

$$r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_P - \left(\frac{A}{2}\right)\sigma_P^2$$

is

$$\frac{\mu_r - r_f}{\sigma_r} - A\sigma_P^* = 0,$$

implying that the optimal choice is

$$\sigma_P^* = \frac{\mu_r - r_f}{A\sigma_r}.$$

2. Portfolio Allocation with Mean-Variance Utility, Part II

Since the portfolio with standard deviation σ_P allocates the fraction

$$w = \frac{\sigma_P}{\sigma_r}$$

of wealth to the risky asset, the solution

$$\sigma_P^* = \frac{\mu_r - r_f}{A\sigma_r}$$

derived above must allocate the share

$$w^* = \frac{\mu_r - r_f}{A\sigma_r^2}$$

to the risky asset. Intuitively, this share decreases if the investor is more risk averse, that is, if A is larger, or if the risky asset has a return that is more volatile, so that σ_r is larger, and increases when the risk premium on the risky asset $\mu_r - r_f$ increases.

3. Portfolio Allocation with Mean-Variance Utility, Part III

The first-order conditions for choosing w_1 and w_2 to maximize

$$(1 - w_1 - w_2)r_f + w_1\mu_1 + w_2\mu_2 - \left(\frac{A}{2}\right)(w_1^2\sigma_1^2 + w_2^2\sigma_2^2).$$

are

$$-r_f + \mu_1 - A\sigma_1^2 w_1^* = 0$$

and

$$-r_f + \mu_2 - A\sigma_2^2 w_2^* = 0.$$

Rearranging each of these two first-order conditions leads to the solutions

$$w_1^* = \frac{\mu_1 - r_f}{A\sigma_1^2}$$

and

$$w_2^* = \frac{\mu_2 - r_f}{A\sigma_2^2},$$

which show that the portfolio share for each risky asset decreases if the investor is more risk averse or if the risky asset has a return that is more volatile, and increases when the risk premium on the risky asset increases.

Interestingly, while these solutions for w_1^* and w_2^* indicate that more risk averse investors allocate smaller fractions of their portfolios to risky assets, the solutions also imply that

$$\frac{w_1^*}{w_2^*} = \frac{\frac{\mu_1 - r_f}{\sigma_1^2}}{\frac{\mu_2 - r_f}{\sigma_2^2}},$$

showing that all investors allocate funds to the two risky assets in the same relative proportion, independently of their levels of risk aversion. This result is also implied by the two-fund or separation theorem of Modern Portfolio Theory, which says that all investors will seek to hold the same portfolio of risky assets, namely the tangency portfolio, and adjust their overall portfolios to suit their own levels of risk aversion by substituting into risk-free assets instead.