Solutions to Problem Set 12

ECON 337901 - Financial Economics — Peter Ireland
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1. The Efficient Frontier

With $\mu_1 = 8$, $\mu_2 = 4$, $\mu_3 = 6$, $\sigma_1 = 8$, $\sigma_2 = 4$, $\sigma_3 = 6$, and $\rho_{12} = \rho_{13} = \rho_{23} = 0$, the weights that minimize the variance of the portfolio's random return, subject to the constraint that the expected return equal 6 percent, solve the problem

$$\max_{w_1,w_2} -64w_1^2 - 16w_2^2 - 36(1 - w_1 - w_2)^2 \text{ subject to } 8w_1 + 4w_2 + 6(1 - w_1 - w_2) = 6.$$

Differentiating the Lagrangian for this problem

$$L(w_1, w_2, \lambda) = -64w_1^2 - 16w_2^2 - 36(1 - w_1 - w_2)^2 + \lambda[8w_1 + 4w_2 + 6(1 - w_1 - w_2) - 6]$$

through first by w_1^* and then by w_2^*, in each case setting the result equal to zero, leads to the first-order conditions

$$-128w_1^* + 72(1 - w_1^* - w_2^*) + \lambda^*(8 - 6) = 0$$

and

$$-32w_2^* + 72(1 - w_1^* - w_2^*) + \lambda^*(4 - 6) = 0.$$

These two first-order conditions combine with the constraint

$$8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6$$

to form a system of three equations in three unknowns: w_1^*, w_2^*, and λ^*.

Although there are many ways of solving this system, one approach is to start with the constraint, which implies

$$\begin{aligned} 8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) &= 6 \\ 8w_1^* + 4w_2^* + 6 - 6w_1^* - 6w_2^* &= 6 \\ 2w_1^* - 2w_2^* &= 0, \end{aligned}$$

or simply

$$w_1^* = w_2^*.$$

Evidently, it is optimal to set w_1^* and w_2^* equal to some common value w^*.

Now substitute $w^* = w_1^* = w_2^*$ into the two first-order conditions, so that they simplify to

$$-128w^* + 72(1 - 2w^*) + 2\lambda^* = 0$$

and

$$-32w^* + 72(1 - 2w^*) - 2\lambda^* = 0.$$

Adding these two equalities works to eliminate λ^* and leaves

$$\begin{aligned} -160w^* + 144(1 - 2w^*) &= 0 \\ -160w^* + 144 - 288w^* &= 0 \\ 144 &= 448w^*, \end{aligned}$$

or

$$w^* = \frac{144}{448} = 0.3214.$$

The optimal portfolio weights are therefore given by

$$w_1^* = w_2^* = w^* = 0.3214$$

and

$$w_3^* = 1 - w_1^* - w_2^* = 1 - 2w^* = 0.3571.$$

That is, the efficient portfolio allocates 0.32 shares each to assets 1 and 2 and the remaining 0.36 share to asset 3.

It now only remains to calculate the minimized standard deviation

$$\sigma_P^* = \sqrt{64(w_1^*)^2 + 16(w_2^*)^2 + 36(w_3^*)^2} = 3.5857.$$

The gains from diversification in this case are quite large, since holding asset 3 alone provides the same expected return of 6 percent but with a standard deviation of 6.