Solutions to Problem Set 12

ECON 337901 - Financial Economics
Boston College, Department of Economics

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1. The Gains From Diversification

With two assets, one with $\mu_1 = 8$ and $\sigma_1 = 8$ and the other with $\mu_2 = 4$ and $\sigma_2 = 4$, and with the share w allocated to asset 1, the expected return of the portfolio is

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

and the standard deviation of the portfolio is

$$\sigma_P = \sqrt{w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}},$$

where ρ_{12} is the correlation between the two returns. Probably, the easiest way to compute μ_P and σ_P for a range of values for w is using a spreadsheet or some other computer program, although the values can also be found with a handheld calculator. In any case, the results are tabulated below.

w	μ_P	σ_P with $\rho_{12} = 0$	σ_P with $\rho_{12} = -0.5$
0.0	4.0	4.00	4.00
0.2	4.8	3.58	2.77
0.4	5.6	4.00	2.88
0.6	6.4	5.06	4.23
0.8	7.2	6.45	6.04
1.0	8.0	8.00	8.00

The table confirms that in both cases there are gains to diversification, in the sense that there are portfolios of the two assets combined that have *both* higher expected returns and lower standard deviations than asset 2 alone. But the table also confirms that the gains from diversification are strongest when the correlation between the two asset returns is negative.

2. The Efficient Frontier

Although we derived the first-order conditions for this optimal portfolio problem in class, allowing for nonzero correlation across the three asset returns, we can re-derive them here for the special case where the asset returns are uncorrelated. Since $\mu_1 = 8$, $\mu_2 = 4$, and $\mu_3 = 6$, the expected return on a portfolio that allocates shares w_1, w_2, and $1 - w_1 - w_2$ to assets 1, 2, and 3 is

$$\mu_P = 8w_1 + 4w_2 + 6(1 - w_1 - w_2) = 6 + 2w_1 - 2w_2.$$

And since $\sigma_1 = 8$, $\sigma_2 = 4$, $\sigma_3 = 6$ and the asset returns are all uncorrelated, the variance of the return on the portfolio is

$$\sigma_P^2 = 64w_1^2 + 16w_2^2 + 36(1 - w_1 - w_2)^2 = 36 - 72w_1 - 72w_2 + 100w_1^2 + 72w_1w_2 + 52w_2^2.$$

Hence, the general problem from class

$$\max_{w_1,w_2} -\sigma_P^2 \text{ subject to } \mu_P = \bar{\mu}$$

can in this case be written more specifically as

$$\max_{w_1,w_2} 72w_1 + 72w_2 - 100w_1^2 - 72w_1w_2 - 52w_2^2 - 36 \text{ subject to } 2w_1 - 2w_2 = 0.$$

The Lagrangian for this problem

$$L = 72w_1 + 72w_2 - 100w_1^2 - 72w_1w_2 - 52w_2^2 - 36 + \lambda(2w_1 - 2w_2)$$

leads to the first-order conditions

$$72 - 200w_1^* - 72w_2^* + 2\lambda^* = 0$$

and

$$72 - 72w_1^* - 104w_2^* - 2\lambda^* = 0$$

which can be combined with the constraint

$$2w_1^* - 2w_2^* = 0$$

to form a system of three linear equations in the three unknowns: w_1^*, w_2^*, and λ^*.

There are many ways of solving this system, but perhaps the easiest starts by observing that the constraint requires that

$$w_1^* = w_2^*,$$

while the two first-order conditions require that

$$72 - 200w_1^* - 72w_2^* + 72 - 72w_1^* - 104w_2^* = 0$$

or, more simply,

$$144 - 272w_1^* - 176w_2^* = 0.$$

These last observations imply that

$$w_1^* = w_2^* = \frac{144}{272 + 176} = 0.3214$$

and hence

$$1 - w_1^* - w_2^* = 0.3571.$$

The efficient portfolio allocates 0.32 shares each to assets 1 and 2 and the remaining 0.36 share to asset 3.

It now only remains to calculate the minimized standard deviation

$$\sigma_P = \sqrt{64w_1^2 + 16w_2^2 + 36(1 - w_1 - w_2)^2} = 3.5857.$$

The gains from diversification in this case are quite large, since holding asset 3 alone provides the same expected return of 6 percent but with a standard deviation of 6.