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FINANCIAL ECONOMICS

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12 The Consumption CAPM

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Key Assumptions

The Consumption Capital Asset Pricing Model was developed by Robert Lucas (US, b.1937, Nobel Prize 1995) in the late 1970s.

Robert Lucas, "Asset Prices in an Exchange Economy," *Econometrica* Vol.46 (November 1978): pp.1429-1445.

The CCAPM specializes the more general Arrow-Debreu model to focus on the pricing of long-lived assets, particularly stocks but also long-term bonds.

Key Assumptions

The CCAPM assumes that all investors are identical in terms of their preferences and endowments.

This assumption allows us to characterize outcomes in financial markets and the economy as a whole by studying the behavior of a single representative consumer/investor.

This assumption can be weakened (generalized) somewhat.

Key Assumptions

If investors have CRRA utility functions with the same coefficient of relative risk aversion or CARA utility functions with possibly different coefficients of absolute risk aversion, they can differ in their endowments. In these cases, their individual consumptions will depend on their wealth levels but their individual marginal rates of substitution will not.

Hence, equilibrium asset prices will not depend on the distribution of wealth. They behave as if they are generated in an economy with a single representative agent.

Key Assumptions

Obviously, the assumption that there is a single representative investor limits the model's usefulness in helping us understand:

1. How investors use financial markets to diversify away idiosyncratic risks.
2. More generally, how differences in preferences – particularly differences in risk aversion – help determine asset prices.

On the other hand, the assumption makes it possible to obtain a sharper view of how equilibrium asset prices reflect aggregate risk.

Key Assumptions

The CCAPM also assumes that investors have infinite horizons.

Thus, if we continue to assume for simplicity that there is a single type of consumption good in each period, and if c_t denotes the representative investor's consumption of that good in each period $t = 0, 1, 2, \ldots$, the investors' preferences are described by the vN-M expected utility function

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

where the discount factor β lies between zero and one.

Key Assumptions

The assumption of infinite horizons is unrealistic if taken literally (remember what Benjamin Franklin said about death and taxes).

But it can be justified by assuming that mortal investors have a bequest motive as suggested by Robert Barro, "Are Government Bonds Net Wealth?" *Journal of Political Economy* Vol.82 (November-December 1974): pp.1095-1117.

Key Assumptions

To illustrate Barro's idea, suppose that each individual from "generation t" cares not only about his or her own lifetime consumption C_t but also about the utility of his or her children, from generation $t+1$. Then

$$V_t = U(C_t) + \delta V_{t+1}$$

where V_t is total utility of generation t and δ measures the strength of the bequest motive.

Key Assumptions

But if members of generation $t+1$ also care about their children

$$V_{t+1} = U(C_{t+1}) + \delta V_{t+2}$$

Similarly,

$$V_{t+2} = U(C_{t+2}) + \delta V_{t+3}$$

and so on forever.

Key Assumptions

In Barro's model

$$V_t = U(C_t) + \delta V_{t+1}$$

$$V_{t+1} = U(C_{t+1}) + \delta V_{t+2}$$

$$V_{t+2} = U(C_{t+2}) + \delta V_{t+3}$$

combine to yield a utility function for a "dynastic family" of the same form

$$V_t = U(C_t) + \delta U(C_{t+1}) + \delta^2 U(C_{t+2}) + \dots$$

assumed by Lucas.

Key Assumptions

Another possible justification for infinite horizons is suggested by Olivier Blanchard, "Debt, Deficits, and Finite Horizons,"' *Journal of Political Economy* Vol.93 (April 1985): pp.223-247.

Blanchard assumed that each consumer is mortal, and faces a small probability p of dying at the beginning of each period t.

Key Assumptions

Hence, each of Blanchard's consumers looks forward from period t and sees that

$$1 - p = \text{probability of living through period } t+1$$

$$(1 - p)^2 = \text{probability of living through period } t+2$$

$$\dots$$

$$(1 - p)^\tau = \text{probability of living through period } t+\tau$$

Assuming that "utility when dead is zero," his or her expected utility from period t forward is

$$U(C_t) + (1-p)U(C_{t+1}) + (1-p)^2 U(C_{t+2}) + \dots$$

again of the same form assumed by Lucas.

Key Assumptions

Blanchard's model is also unrealistic, since it implies that each person has a very small probability of living 200 years or more.

But what his model highlights is that the real reason for assuming infinite horizons is to avoid the "time $T-1$" problem: if everyone knows the world will end at T, no one is going to buy stocks at $T-1$. But, knowing this makes stocks less attractive at $T-2$ asfor well. The collapse in stock prices will start before the terminal date.

The infinite horizon prevents this unraveling.

Key Assumptions

Obviously, the assumption that investors have infinite horizons limit's the model's usefulness in helping us understand "life-cycle" behavior such as:

1. Borrowing to pay for college or a house.
2. Saving for retirement.

On the other hand, it eliminates a mathematical curiosity that would otherwise influence the prices of long-lived assets in the model.

Lucas' Tree Model

Lucas imagined an economy in which the only source of consumption is the fruit that grows on trees.

Individual consumers/investors buy and sell prices of fruit and shares in each tree at each date $t = 0, 1, 2, \ldots$.

Lucas' Tree Model

Let Y_t denote the number of pieces of fruit produced by each tree during period t.

Let z_t denote the number of shares held by the representative investor at the beginning of period t. Then z_{t+1} is the number of shares purchased by the representative investor during t and carried into $t + 1$.

Let P_t denote the price of each share in a tree during period t, measured in units of the consumption good (consumption is the numeraire).

Lucas' Tree Model

The investor's z_t shares of trees held at the beginning of period t entitles him or her to $Y_t z_t$ pieces of fruit, grown on those shares of the trees.

Thus, during each period $t = 0, 1, 2, \ldots$, the representative investor faces the budget constraint

$$P_t z_t + Y_t z_t \geq c_t + P_t z_{t+1}$$

Lucas' Tree Model

Hence, in Lucas' Tree Model:

1. Shares in trees are like shares of stock.
2. The fruit that growth on trees become the dividends paid by shares of stock.

The simplified story makes clear that

1. The value of all shares of stock measures the value of an economy's productive assets.
2. The dividends paid by stock reflects the flow of output produced by those assets.

thereby drawing on the A-D model's ability to link financial markets back to the economy as a whole.

Lucas' Tree Model

In Lucas' model, dividends take on one of N possible values in each period:

$$Y_t \in \{Y^1, Y^2, \dots, Y^N\}$$

The randomness in dividends is governed by Markov chain, named after Andrey Markov (Russia, 1856-1922).

In a Markov chain, the probabilities for dividends at $t+1$ are allowed to depend on the outcome for dividends at t, but not on the outcome for dividends in periods before t.

Lucas' Tree Model

With dividends governed by a Markov chain:

$$\pi_{ij} = \text{Prob}(Y_{t+1} = Y^j | Y_t = Y^i)$$

This allows for serial correlation in dividends: high dividends this year may be more likely followed by high dividends next year and low dividends this year may be more likely followed by low dividends next year.

Lucas' Tree Model

Hence, faced with uncertainty about future dividends, the representative investor in the Tree Model chooses how much to consume c_t and how many shares to buy z_{t+1} in each period $t = 0, 1, 2, \dots$ to maximize the vN-M expected utility function

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

subject to the budget constraint

$$P_t z_t + Y_t z_t \geq c_t + P_t z_{t+1}$$

Lucas' Tree Model

This optimization problem is explicitly

1. Dynamic - choices get made at different points in time
2. Stochastic - choices at t get made knowing past dividends, but viewing future dividends as random

Dynamic Programming methods for solving dynamic, stochastic optimization problems were developed in the late 1950s by Richard Bellman and require a heavy investment in probability theory as well mathematical analysis.

Lucas' Tree Model

To derive the key optimality condition heuristically, let c_t^* and z_{t+1}^* be the values that solve the investor's problem, and consider a deviation from these optimal choices that involves consuming slightly less at t

$$c_t = c_t^* - \varepsilon$$

using the extra amount ε saved to buy ε/P_t more shares at t

$$z_{t+1} = z_{t+1}^* + \varepsilon/P_t$$

then collecting the dividends and selling the extra shares to consume more at $t+1$

$$c_{t+1} = c_{t+1}^* + (\varepsilon/P_t)(Y_{t+1} + P_{t+1})$$

Lucas' Tree Model

When this deviation is considered at t, it lowers utility at t but raises expected utility at $t+1$ according to

$$u(c_t^* - \varepsilon) + \beta E_t\{u[c_{t+1}^* + (\varepsilon/P_t)(Y_{t+1} + P_{t+1})]\}$$

where

$$E_t = \text{ expected value in period } t$$

reflects the fact when decisions are made at t, the values of $Y_t, Y_{t-1}, Y_{t-2}, \ldots$ are known but the values of $Y_{t+1}, Y_{t+2}, Y_{t+3}, \ldots$ are still random.

Lucas' Tree Model

Using

$$u(c_t^* - \varepsilon) + \beta E_t\{u[c_{t+1}^* + (\varepsilon/P_t)(Y_{t+1} + P_{t+1})]\}$$

the "first-order condition" for the optimal ε is

$$\begin{aligned} 0 &= -u'(c_t^* - \varepsilon^*) \\ &\quad + \beta E_t\left\{u'\left[c_{t+1}^* + \left(\frac{\varepsilon^*}{P_t}\right)(Y_{t+1} + P_{t+1})\right]\left(\frac{Y_{t+1} + P_{t+1}}{P_t}\right)\right\} \end{aligned}$$

Lucas' Tree Model

But if c_t^* and c_{t+1}^* are really the optimal choices, ε^* must equal zero, so that

$$\begin{aligned} 0 &= -u'(c_t^* - \varepsilon^*) \\ &\quad + \beta E_t \left\{ u' \left[c_{t+1}^* + \left(\frac{\varepsilon^*}{P_t} \right) (Y_{t+1} + P_{t+1}) \right] \left(\frac{Y_{t+1} + P_{t+1}}{P_t} \right) \right\} \end{aligned}$$

implies

$$u'(c_t^*) = \beta E_t \left[u'(c_{t+1}^*) \left(\frac{Y_{t+1} + P_{t+1}}{P_t} \right) \right]$$

Lucas' Tree Model

But

$$u'(c_t^*) = \beta E_t \left[u'(c_{t+1}^*) \left(\frac{Y_{t+1} + P_{t+1}}{P_t} \right) \right]$$

is just another version of the Euler equation we derived previously, since the random return on a share purchased at t and sold after collecting the dividends at $t+1$ is

$$R_{t+1} = \frac{Y_{t+1} + P_{t+1}}{P_t}$$

Lucas' Tree Model

In the Tree Model, as in the more general A-D model, the Euler equation

$$u'(c_t) = \beta E_t \left[u'(c_{t+1}) \left(\frac{Y_{t+1} + P_{t+1}}{P_t} \right) \right]$$

describing the investor's optimal choices gets combined with market clearing conditions for shares and fruit to explicitly link asset prices to developments in the economy as a whole.

Lucas' Tree Model

Assume that is one tree per consumer/investor in the economy as a whole.

Then, in a competitive equilibrium, prices must adjust so that

$$z_t = z_{t+1} = 1 \text{ and } c_t = Y_t$$

for all $t = 0, 1, 2, \ldots$.

The representative investor must willing holds all the shares in and consume all of the fruit from "his or her tree."

Lucas' Tree Model

Hence, in equilibrium, the Euler equation

$$u'(c_t) = \beta E_t \left[u'(c_{t+1}) \left(\frac{Y_{t+1} + P_{t+1}}{P_t} \right) \right]$$

implies

$$u'(Y_t) = \beta E_t \left[u'(Y_{t+1}) \left(\frac{Y_{t+1} + P_{t+1}}{P_t} \right) \right]$$

Lucas' Tree Model

Rewrite the equilibrium condition

$$u'(Y_t) = \beta E_t \left[u'(Y_{t+1}) \left(\frac{Y_{t+1} + P_{t+1}}{P_t} \right) \right]$$

as

$$u'(Y_t)P_t = \beta E_t[u'(Y_{t+1})Y_{t+1}] + \beta E_t[u'(Y_{t+1})P_{t+1}]$$

and consider the same condition, one for period later:

$$u'(Y_{t+1})P_{t+1} = \beta E_{t+1}[u'(Y_{t+2})Y_{t+2}] + \beta E_{t+1}[u'(Y_{t+2})P_{t+2}]$$

Lucas' Tree Model

To usefully combine these conditions, we need to rely on a result from statistical theory, the law of iterated expectations.

For a random variable X_{t+2} that becomes known at time $t+2$:

$$E_t[E_{t+1}(X_{t+2})] = E_t(X_{t+2}).$$

In words: "my expectation today of my expectation next year of stock prices two years from now" should be the same the same as "my expectation today of stock prices two years from now."

Lucas' Tree Model

Substitute

$$u'(Y_{t+1})P_{t+1} = \beta E_{t+1}[u'(Y_{t+2})Y_{t+2}] + \beta E_{t+1}[u'(Y_{t+2})P_{t+2}]$$

into

$$u'(Y_t)P_t = \beta E_t[u'(Y_{t+1})Y_{t+1}] + \beta E_t[u'(Y_{t+1})P_{t+1}]$$

and use the law of iterated expectations to obtain

$$\begin{aligned} u'(Y_t)P_t &= \beta E_t[u'(Y_{t+1})Y_{t+1}] \\ &\quad + \beta^2 E_t[u'(Y_{t+2})Y_{t+2}] + \beta^2 E_t[u'(Y_{t+2})P_{t+2}] \end{aligned}$$

Lucas' Tree Model

$$
\begin{aligned}
u'(Y_t)P_t &= \beta E_t[u'(Y_{t+1})Y_{t+1}] \\
&\quad + \beta^2 E_t[u'(Y_{t+2})Y_{t+2}] + \beta^2 E_t[u'(Y_{t+2})P_{t+2}]
\end{aligned}
$$

Continuing in this manner using

$$u'(Y_{t+2})P_{t+2} = \beta E_{t+2}[u'(Y_{t+3})Y_{t+3}] + \beta E_{t+2}[u'(Y_{t+3})P_{t+3}]$$

eventually yields

$$u'(Y_t)P_t = E_t\left[\sum_{\tau=1}^{\infty} \beta^\tau u'(Y_{t+\tau})Y_{t+\tau}\right]$$

Lucas' Tree Model

$$u'(Y_t)P_t = E_t\left[\sum_{\tau=1}^{\infty} \beta^{\tau} u'(Y_{t+\tau}) Y_{t+\tau}\right]$$

rewritten as

$$P_t = E_t\left\{\sum_{\tau=1}^{\infty} \left[\frac{\beta^{\tau} u'(Y_{t+\tau})}{u'(c_t)}\right] Y_{t+\tau}\right\}$$

indicates that in the Tree Model, "the price of a stock equals the present discounted value of all the future dividends," where the discount factor is given by the representative investor's intertemporal marginal rate of substitution.

Lucas' Tree Model

To obtain more specific results, suppose that the representative investor's Bernoulli utility function is of the CRRA form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

and that dividends can take on three possible values

$$Y_t \in \{Y^1, Y^2, Y^3\} = \{0.5, 1.0, 1.5\}$$

with

$$\pi_{ij} = \begin{cases} 0.50 & \text{if } j = i \\ 0.25 & \text{if } j \neq i \end{cases}$$

so that they display some "inertia."

Lucas' Tree Model

With CRRA utility, $u'(Y) = Y^{-\gamma}$, so that

$$u'(Y_t)P_t = \beta E_t[u'(Y_{t+1})(Y_{t+1} + P_{t+1})]$$

implies

$$Y_t^{-\gamma} P_t = \beta E_t(Y_{t+1}^{1-\gamma} + Y_{t+1}^{-\gamma} P_{t+1})$$

Lucas' Tree Model

Let P^1, P^2, and P^3 be the share prices when Y_t equals Y^1, Y^2, and Y^3. Then when $Y_t = Y^1$,

$$Y_t^{-\gamma} P_t = \beta E_t (Y_{t+1}^{1-\gamma} + Y_{t+1}^{-\gamma} P_{t+1})$$

implies

$$\begin{aligned}(Y^1)^{-\gamma} P^1 \quad = \quad & \beta\pi_{11}[(Y^1)^{1-\gamma} + (Y^1)^{-\gamma} P^1] \\ & + \beta\pi_{12}[(Y^2)^{1-\gamma} + (Y^2)^{-\gamma} P^2] \\ & + \beta\pi_{13}[(Y^3)^{1-\gamma} + (Y^3)^{-\gamma} P^3]\end{aligned}$$

Lucas' Tree Model

Similarly, when $Y_t = Y^2$

$$\begin{aligned}
(Y^2)^{-\gamma} P^2 &= \beta\pi_{21}[(Y^1)^{1-\gamma} + (Y^1)^{-\gamma} P^1] \\
&+ \beta\pi_{22}[(Y^2)^{1-\gamma} + (Y^2)^{-\gamma} P^2] \\
&+ \beta\pi_{23}[(Y^3)^{1-\gamma} + (Y^3)^{-\gamma} P^3]
\end{aligned}$$

and when $Y_t = Y^3$

$$\begin{aligned}
(Y^3)^{-\gamma} P^3 &= \beta\pi_{31}[(Y^1)^{1-\gamma} + (Y^1)^{-\gamma} P^1] \\
&+ \beta\pi_{32}[(Y^2)^{1-\gamma} + (Y^2)^{-\gamma} P^2] \\
&+ \beta\pi_{33}[(Y^3)^{1-\gamma} + (Y^3)^{-\gamma} P^3]
\end{aligned}$$

Lucas' Tree Model

Plug in the specific values for dividends and probabilities ...

$$
\begin{aligned}
(0.5)^{-\gamma}P^1 &= \beta 0.50[(0.5)^{1-\gamma} + (0.5)^{-\gamma}P^1] \\
&\quad + \beta 0.25[(1.0)^{1-\gamma} + (1.0)^{-\gamma}P^2] \\
&\quad + \beta 0.25[(1.5)^{1-\gamma} + (1.5)^{-\gamma}P^3]
\end{aligned}
$$

$$
\begin{aligned}
(1.0)^{-\gamma}P^2 &= \beta 0.25[(0.5)^{1-\gamma} + (0.5)^{-\gamma}P^1] \\
&\quad + \beta 0.50[(1.0)^{1-\gamma} + (1.0)^{-\gamma}P^2] \\
&\quad + \beta 0.25[(1.5)^{1-\gamma} + (1.5)^{-\gamma}P^3]
\end{aligned}
$$

$$
\begin{aligned}
(1.5)^{-\gamma}P^3 &= \beta 0.25[(0.5)^{1-\gamma} + (0.5)^{-\gamma}P^1] \\
&\quad + \beta 0.25[(1.0)^{1-\gamma} + (1.0)^{-\gamma}P^2] \\
&\quad + \beta 0.50[(1.5)^{1-\gamma} + (1.5)^{-\gamma}P^3]
\end{aligned}
$$

Lucas' Tree Model

. . . to obtain a set of 3 equations in 3 unknowns . . .

$$
\begin{aligned}
(0.5)^{-\gamma}P^1 &= \beta 0.50[(0.5)^{1-\gamma} + (0.5)^{-\gamma}P^1] \\
&\quad + \beta 0.25[(1.0)^{1-\gamma} + (1.0)^{-\gamma}P^2] \\
&\quad + \beta 0.25[(1.5)^{1-\gamma} + (1.5)^{-\gamma}P^3]
\end{aligned}
$$

$$
\begin{aligned}
(1.0)^{-\gamma}P^2 &= \beta 0.25[(0.5)^{1-\gamma} + (0.5)^{-\gamma}P^1] \\
&\quad + \beta 0.50[(1.0)^{1-\gamma} + (1.0)^{-\gamma}P^2] \\
&\quad + \beta 0.25[(1.5)^{1-\gamma} + (1.5)^{-\gamma}P^3]
\end{aligned}
$$

$$
\begin{aligned}
(1.5)^{-\gamma}P^3 &= \beta 0.25[(0.5)^{1-\gamma} + (0.5)^{-\gamma}P^1] \\
&\quad + \beta 0.25[(1.0)^{1-\gamma} + (1.0)^{-\gamma}P^2] \\
&\quad + \beta 0.50[(1.5)^{1-\gamma} + (1.5)^{-\gamma}P^3]
\end{aligned}
$$

Lucas' Tree Model

... that is linear in P^1, P^2, P^3

$$
\begin{aligned}
(0.5)^{-\gamma} P^1 &= \beta 0.50[(0.5)^{1-\gamma} + (0.5)^{-\gamma} P^1] \\
&\quad + \beta 0.25[(1.0)^{1-\gamma} + (1.0)^{-\gamma} P^2] \\
&\quad + \beta 0.25[(1.5)^{1-\gamma} + (1.5)^{-\gamma} P^3]
\end{aligned}
$$

$$
\begin{aligned}
(1.0)^{-\gamma} P^2 &= \beta 0.25[(0.5)^{1-\gamma} + (0.5)^{-\gamma} P^1] \\
&\quad + \beta 0.50[(1.0)^{1-\gamma} + (1.0)^{-\gamma} P^2] \\
&\quad + \beta 0.25[(1.5)^{1-\gamma} + (1.5)^{-\gamma} P^3]
\end{aligned}
$$

$$
\begin{aligned}
(1.5)^{-\gamma} P^3 &= \beta 0.25[(0.5)^{1-\gamma} + (0.5)^{-\gamma} P^1] \\
&\quad + \beta 0.25[(1.0)^{1-\gamma} + (1.0)^{-\gamma} P^2] \\
&\quad + \beta 0.50[(1.5)^{1-\gamma} + (1.5)^{-\gamma} P^3]
\end{aligned}
$$

Lucas' Tree Model

With $\beta = 0.96$ and $\{Y^1, Y^2, Y^3\} = \{0.5, 1.0, 1.5\}$

γ	P^1	P^2	P^3
0.5	16.5	23.5	28.8
1.0	12.0	24.0	36.0
2.0	7.4	29.3	65.6

Stock prices are "procyclical" and become more volatile as the coefficient of relative risk aversion increases.

Deriving the CCAPM

Although the Tree Model assumes there is only one asset, we can turn it into a more general model by introducing additional assets.

Let $R_{j,t+1}$ denote the gross return on asset j between t and $t+1$, and let $r_{j,t+1}$ be the associated net return, so that

$$1 + r_{j,t+1} = R_{j,t+j}$$

Deriving the CCAPM

For shares in the tree, the gross return

$$R_{t+1} = \frac{Y_{t+1} + P_{t+1}}{P_t}$$

and the net return

$$r_{t+1} = \frac{Y_{t+1} + P_{t+1} - P_t}{P_t}$$

account for both the dividend Y_{t+1} and the capital gain or loss $P_{t+1} - P_t$.

Deriving the CCAPM

More generally, the Euler equation implies that the return on any asset j must satisfy

$$u'(c_t) = \beta E_t[u'(c_{t+1})R_{j,t+1}] = \beta E_t[u'(c_{t+1})(1 + r_{j,t+1})]$$

where, now, the representative investor's consumption c_t includes income from all assets and possibly labor as well.

Deriving the CCAPM

Consider first a riskless asset, like a bank account or a short-term Government bond, with return $r_{f,t+1}$ that is known at t. For this asset, the Euler equation

$$u'(c_t) = \beta E_t[u'(c_{t+1})R_{j,t+1}] = \beta E_t[u'(c_{t+1})(1 + r_{f,t+1})]$$

implies

$$\frac{1}{1 + r_{f,t+1}} = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]$$

Remember: This condition generalizes Irving Fisher's theory of interest to the case where randomness in other asset returns introduces randomness into future consumption as well.

Deriving the CCAPM

Next, consider a risky asset. The Euler equation

$$u'(c_t) = \beta E_t[u'(c_{t+1})R_{j,t+1}] = \beta E_t[u'(c_{t+1})(1 + r_{j,t+1})]$$

can be written equivalently as

$$1 = E_t \left\{ \left[\frac{\beta u'(c_{t+1})}{u'(c_t)} \right] (1 + r_{j,t+1}) \right\}$$

But what does this equation imply about $E_t r_{j,t+1}$, the expected return on the risky asset?

Deriving the CCAPM

Recall that for any two random variables X and Y with $E(X) = \mu_X$ and $E(Y) = \mu_Y$, the covariance between X and Y is defined as

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

This definition implies

$$\begin{aligned}
Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\
&= E(XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y) \\
&= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X \mu_Y \\
&= E(XY) - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y \\
&= E(XY) - E(X)E(Y)
\end{aligned}$$

Deriving the CCAPM

Since

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

or

$$E(XY) = E(X)E(Y) + Cov(X, Y)$$

The Euler equation

$$1 = E_t \left\{ \left[\frac{\beta u'(c_{t+1})}{u'(c_t)} \right] (1 + r_{j,t+1}) \right\}$$

implies

$$1 = E_t \left[\frac{\beta u'(c_{t+1})}{u'(c_t)} \right] E_t(1 + r_{j,t+1}) + Cov_t \left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1} \right]$$

Deriving the CCAPM

Combine

$$1 = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right] E_t(1 + r_{j,t+1}) + Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

with

$$\frac{1}{1 + r_{f,t+1}} = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]$$

to obtain

$$1 = \frac{E_t(1 + r_{j,t+1})}{1 + r_{f,t+1}} + Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

Deriving the CCAPM

$$1 = \frac{E_t(1 + r_{j,t+1})}{1 + r_{f,t+1}} + Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

implies

$$1 + r_{f,t+1} = 1 + E_t(r_{j,t+1}) + (1 + r_{f,t+1})Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

and hence

$$E_t(r_{j,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

Deriving the CCAPM

$$E_t(r_{j,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1}) Cov_t \left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1} \right]$$

This equation is beginning to look like the equations from the CAPM.

In fact, it has similar implications.

Deriving the CCAPM

$$E_t(r_{j,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

The expected return on asset j will be above the risk-free rate if the covariance between the actual return on asset j and the representative investor's IMRS is negative.

Deriving the CCAPM

If u is concave, the investor's IMRS

$$\frac{\beta u'(c_{t+1})}{u'(c_t)}$$

will be high if c_{t+1} is low relative to c_t and low if c_{t+1} is high relative to c_t.

Hence the IMRS is inversely related to the business cycle: it is high during recessions and low during booms.

Deriving the CCAPM

$$E_t(r_{j,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

The expected return on asset j will be above the risk-free rate if the covariance between the actual return on asset j and the representative investor's IMRS is negative – that is, if the asset return is high during booms and low during recessions.

This asset exposes investors to additional aggregate risk. In equilibrium, it must offer a higher expected return to compensate.

Deriving the CCAPM

$$E_t(r_{j,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

Conversely, the expected return on asset j will be below the risk-free rate if the covariance between the actual return on asset j and the representative investor's IMRS is positive – that is, if the asset return is high during recessions and low during booms.

This asset insures investors against aggregate risk. Its low expected return reflects the premium that investors are willing to pay to obtain this insurance.

Deriving the CCAPM

$$E_t(r_{j,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1}) Cov_t \left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1} \right]$$

Like the traditional CAPM, the CCAPM implies that assets offer higher expected returns only when they expose investors to additional aggregate risk. The CCAPM goes further, by explicitly linking aggregate risk to the business cycle.

Deriving the CCAPM

To draw even closer connections between the CCAPM and the traditional CAPM, suppose now that there is an asset with random return $R_{c,t+1} = 1 + r_{c,t+1}$ that coincides with the representative investor's IMRS:

$$R_{c,t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$

Note that this asset has a high return when the IMRS is high, that is, during a recession.

Deriving the CCAPM

Applying the general formula

$$E_t(r_{j,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

to this asset yields

$$E_t(r_{c,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, \frac{\beta u'(c_{t+1})}{u'(c_t)}\right]$$

$$E_t(r_{c,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Var_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]$$

Deriving the CCAPM

$$E_t(r_{c,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Var_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]$$

can be rewritten as

$$-(1 + r_{f,t+1}) = \frac{E_t(r_{c,t+1}) - r_{f,t+1}}{Var_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]}$$

and substituted into the more general equation

$$E_t(r_{j,t+1}) - r_{f,t+1} = -(1 + r_{f,t+1})Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]$$

Deriving the CCAPM

$$E_t(r_{j,t+1}) - r_{f,t+1} = \frac{Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]}{Var_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]}[E_t(r_{c,t+1}) - r_{f,t+1}]$$

But note that

$$\beta_{j,c} = \frac{Cov_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}, r_{j,t+1}\right]}{Var_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right]}$$

is the slope coefficient from a regression of $r_{j,t+1}$ on the IMRS and therefore analogous to beta from the traditional CAPM.

Deriving the CCAPM

Hence, the implications of the CCAPM can be summarized by

$$E_t(r_{j,t+1}) - r_{f,t+1} = \beta_{j,c}[E_t(r_{c,t+1}) - r_{f,t+1}]$$

where $\beta_{j,c}$ and $r_{c,t+1}$ refer to the representative investors IMRS instead of the return on the CAPM's market portfolio.

Both theories indicate that the market will only compensate investors with higher expected returns when they purchase assets that expose them to additional aggregate risk.

Deriving the CCAPM

In the end, therefore, the CAPM and CCAPM deliver a similar message, but differ in how they summarize or measure aggregate risk.

The CAPM measures exposure to aggregate risk using the correlation with the return on the market portfolio.

The CCAPM measures exposure to aggregate risk using the correlation with the IMRS and, ultimately, consumption.

Testing the CCAPM

A famous paper that evaluated CCAPM in terms of its ability to account for average returns on stocks and bonds in the US is Rajnish Mehra and Edward Prescott, "The Equity Premium: A Puzzle," *Journal of Monetary Economics* Vol.15 (March 1985): pp.145-161.

Edward Prescott (US. b.1940) won the Nobel Prize in 2004.

Testing the CCAPM

Mehra and Prescott's results are strikingly negative, in that they show that the CCAPM has great difficulty matching even the most basic aspects of the data.

But their paper has inspired an enormous amount of additional research, which continues today, directed at modifying or extending the model to improves its empirical performance.

Testing the CCAPM

To compare the CCAPM's predictions to US data, Mehra and Prescott began by modifying Lucas' Tree Model to allow for fluctuations in consumption growth as opposed to consumption itself, reflecting the fact that in the US, consumption follows an upward trend over time.

But they continued to assume that there is a single representative investor with an infinite horizon and CRRA utility:

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

Testing the CCAPM

We've already seen that with these preferences, the investor's Euler equation and the equilibrium condition $c_t = Y_t$ imply

$$u'(Y_t)P_t = \beta E_t[u'(Y_{t+1})(Y_{t+1} + P_{t+1})]$$

$$Y_t^{-\gamma} P_t = \beta E_t[Y_{t+1}^{-\gamma}(Y_{t+1} + P_{t+1})]$$

$$P_t = \beta E_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\gamma} (Y_{t+1} + P_{t+1}) \right]$$

$$P_t = \beta E_t[G_{t+1}^{-\gamma}(Y_{t+1} + P_{t+1})]$$

where $G_{t+1} = Y_{t+1}/Y_t$ is the gross rate of consumption growth between t and $t+1$.

Testing the CCAPM

Mehra and Prescott assumed that consumption growth G_{t+1} is log-nomally distributed, meaning that the natural logarithm of G_{t+1} is normally distributed, with

$$\ln(G_{t+1}) \sim N(\mu_g, \sigma_g^2)$$

They also assumed that G_{t+1} is independent and identically distributed (iid) over time, so that the mean μ_g and variance σ_g^2 of the log of G_{t+1} are constant over time.

Testing the CCAPM

Let $g_{t+1} = G_{t+1} - 1$ denote the net rate of consumption growth.

The approximation

$$\ln(G_{t+1}) = \ln(1 + g_{t+1}) \approx g_{t+1}$$

shows that since G_{t+1} is log-normally distributed, $\ln(G_{t+1})$ is normally distributed, and therefore g_{t+1} is approximately normally distributed.

Testing the CCAPM

Since, by definition,

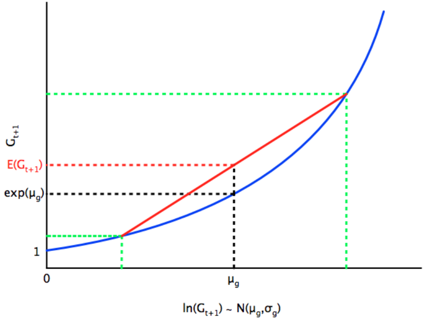
$$G_{t+1} = \exp[\ln(G_{t+1})]$$

where $\exp(x) = e^x$ denotes the exponential function, Jensen's inequality implies that the mean and variance of G_{t+1} can't be found simply by calculating $\exp(\mu_g)$ and $\exp(\sigma_g^2)$.

In particular, since the exponential function is convex

$$E(G_{t+1}) > \exp\{E[\ln(G_{t+1})]\} = \exp(\mu_g)$$

Testing the CCAPM



Jensen's inequality implies that $E(G_{t+1}) > \exp(\mu_g)$, where $\mu_g = E[ln(G_{t+1})]$.

Testing the CCAPM

In particular, if G_{t+1} is log-normally distributed, with

$$\ln(G_{t+1}) \sim N(\mu_g, \sigma_g^2)$$

then

$$E(G_{t+1}) = \exp\left(\mu_g + \frac{1}{2}\sigma_g^2\right)$$

where the $(1/2)\sigma_g^2$ is the "Jensen's inequality term." In addition

$$E(G_{t+1}^{\alpha}) = \exp\left(\alpha\mu_g + \frac{1}{2}\alpha^2\sigma_g^2\right)$$

for any value of α.

Testing the CCAPM

In general, the Euler equation

$$u'(Y_t)P_t = \beta E_t[u'(Y_{t+1})(Y_{t+1} + P_{t+1})]$$

has a mathematical structure similar to that of a differential equation.

With CRRA utility and iid consumption growth, a "guess-and-verify" procedure similar to those used to solve many differential equations can be used to find the solution for P_t in terms of Y_t and P_{t+1} in terms of Y_{t+1}.

Testing the CCAPM

Suppose, in particular, that

$$P_t = vY_t \text{ and } P_{t+1} = vY_{t+1}$$

where v is a constant, to be determined.

Substitute these guesses into the Euler equation

$$P_t = \beta E_t[G_{t+1}^{-\gamma}(Y_{t+1} + P_{t+1})]$$

to obtain

$$vY_t = \beta E_t[G_{t+1}^{-\gamma}(Y_{t+1} + vY_{t+1})]$$

Testing the CCAPM

$$vY_t = \beta E_t[G_{t+1}^{-\gamma}(Y_{t+1} + vY_{t+1})]$$

implies

$$v = \beta E_t\left[G_{t+1}^{-\gamma}(1+v)\left(\frac{Y_{t+1}}{Y_t}\right)\right]$$

$$v = (1+v)\beta E_t(G_{t+1}^{1-\gamma})$$

and hence

$$v = \frac{\beta E_t(G_{t+1}^{1-\gamma})}{1 - \beta E_t(G_{t+1}^{1-\gamma})}$$

which is constant since $E_t(G_{t+1}^{1-\gamma})$ is constant over time when G_{t+1} is iid.

Testing the CCAPM

Now we are ready to address the question of how well the CCAPM "fits the facts."

Consider, first, the risk-free rate of return $r_{f,t+1}$, which satisfies

$$1 = \beta E_t[G_{t+1}^{-\gamma}(1 + r_{f,t+1})]$$

or

$$1 + r_{f,t+1} = \frac{1}{\beta E_t(G_{t+1}^{-\gamma})}$$

Testing the CCAPM

$$1 + r_{f,t+1} = \frac{1}{\beta E_t(G_{t+1}^{-\gamma})}$$

Since $\ln(G_{t+1}) \sim N(\mu_g, \sigma_g^2)$,

$$E(G_{t+1}^{\alpha}) = \exp\left(\alpha\mu_g + \frac{1}{2}\alpha^2\sigma_g^2\right)$$

for any value of α. In particular,

$$E(G_{t+1}^{-\gamma}) = \exp\left(-\gamma\mu_g + \frac{1}{2}\gamma^2\sigma_g^2\right)$$

Testing the CCAPM

$$E(G_{t+1}^{-\gamma}) = \exp\left(-\gamma\mu_g + \frac{1}{2}\gamma^2\sigma_g^2\right)$$

Now use the fact that

$$\frac{1}{\exp(x)} = \frac{1}{e^x} = e^{-x} = \exp(-x)$$

to rewrite this last equation as

$$\frac{1}{E(G_{t+1}^{-\gamma})} = \exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right)$$

Testing the CCAPM

Substitute

$$\frac{1}{E(G_{t+1}^{-\gamma})} = \exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right)$$

into

$$1 + r_{f,t+1} = \frac{1}{\beta E_t(G_{t+1}^{-\gamma})}$$

to obtain

$$1 + r_{f,t+1} = \left(\frac{1}{\beta}\right)\exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right)$$

Testing the CCAPM

$$1 + r_{f,t+1} = \left(\frac{1}{\beta}\right) \exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right)$$

This equation shows specifically how, according to the model, the risk-free rate depends on the preference parameters β and γ and the mean and variance μ_g and σ_g^2 of log consumption growth.

Testing the CCAPM

Consider, next, the return $r_{e,t+1}$ on stocks (equities), which the CCAPM associates with the return on trees:

$$1 + r_{e,t+1} = \frac{Y_{t+1} + P_{t+1}}{P_t} = \frac{Y_{t+1} + vY_{t+1}}{vY_t} = \left(\frac{1}{v} + 1\right) G_{t+1}$$

implies

$$E_t(r_{e,t+1}) = \left(\frac{1}{v} + 1\right) E_t(G_{t+1}) - 1$$

Testing the CCAPM

$$v = \frac{\beta E_t(G_{t+1}^{1-\gamma})}{1 - \beta E_t(G_{t+1}^{1-\gamma})}$$

implies

$$\frac{1}{v} + 1 = \frac{1 - \beta E_t(G_{t+1}^{1-\gamma})}{\beta E_t(G_{t+1}^{1-\gamma})} + 1 = \frac{1}{\beta E_t(G_{t+1}^{1-\gamma})}$$

and hence

$$E_t(r_{e,t+1}) = \left(\frac{1}{v} + 1\right) E_t(G_{t+1}) - 1$$

implies

$$1 + E_t(r_{e,t+1}) = \frac{E_t(G_{t+1})}{\beta E_t(G_{t+1}^{1-\gamma})}$$

Testing the CCAPM

$$1 + E_t(r_{e,t+1}) = \frac{E_t(G_{t+1})}{\beta E_t(G_{t+1}^{1-\gamma})}$$

Since $\ln(G_{t+1}) \sim N(\mu_g, \sigma_g^2)$,

$$E(G_{t+1}) = \exp\left(\mu_g + \frac{1}{2}\sigma_g^2\right)$$

and

$$E(G_{t+1}^{1-\gamma}) = \exp\left[(1-\gamma)\mu_g + \frac{1}{2}(1-\gamma)^2\sigma_g^2\right]$$

Testing the CCAPM

Therefore

$$
\begin{aligned}
1 + E_t(r_{e,t+1}) &= \frac{E_t(G_{t+1})}{\beta E_t(G_{t+1}^{1-\gamma})} \\
&= \frac{\exp\left(\mu_g + \frac{1}{2}\sigma_g^2\right)}{\beta \exp\left[(1-\gamma)\mu_g + \frac{1}{2}(1-\gamma)^2\sigma_g^2\right]} \\
&= \left(\frac{1}{\beta}\right) \exp\left(\mu_g + \frac{1}{2}\sigma_g^2\right) \\
&\quad \times \exp\left[-(1-\gamma)\mu_g - \frac{1}{2}(1-\gamma)^2\sigma_g^2\right]
\end{aligned}
$$

Testing the CCAPM

Using $e^x e^y = e^{x+y}$

$$\begin{aligned} 1 + E_t(r_{e,t+1}) &= \left(\frac{1}{\beta}\right) \exp\left(\mu_g + \frac{1}{2}\sigma_g^2\right) \\ &\quad \times \exp\left[-(1-\gamma)\mu_g - \frac{1}{2}(1-\gamma)^2\sigma_g^2\right] \end{aligned}$$

simplifies to

$$1 + E_t(r_{e,t+1}) = \left(\frac{1}{\beta}\right) \exp\left(\gamma\mu_g + \frac{1}{2}\gamma^2\sigma_g^2\right) \exp\left(\gamma\sigma_g^2\right)$$

Testing the CCAPM

$$1 + E_t(r_{e,t+1}) = \left(\frac{1}{\beta}\right) \exp\left(\gamma\mu_g + \frac{1}{2}\gamma^2\sigma_g^2\right) \exp\left(\gamma\sigma_g^2\right)$$

to interpret this last result, recall that

$$1 + r_{f,t+1} = \left(\frac{1}{\beta}\right) \exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right)$$

Hence, the two solutions can be combined to obtain something much simpler:

$$1 + E_t(r_{e,t+1}) = (1 + r_{f,t+1}) \exp\left(\gamma\sigma_g^2\right)$$

Testing the CCAPM

Since

$$1 + E_t(r_{e,t+1}) = (1 + r_{f,t+1}) \exp\left(\gamma\sigma_g^2\right)$$

implies

$$\frac{1 + E_t(r_{e,t+1})}{1 + r_{f,t+1}} = \exp\left(\gamma\sigma_g^2\right) > 1$$

Thus, with CRRA utility and iid, log-normal consumption growth, the CCAPM implies an equity premium $E(r_{e,t+1}) - r_{f,t+1}$ that is positive and gets larger as either

1. σ_g^2 increases, so that aggregate risk increases
2. γ increases, so that investors become more risk averse

Testing the CCAPM

Thus, with CRRA utility and iid, log-normal consumption growth, the CCAPM implies an equity premium $r_{e,t+1} - r_{f,t+1}$ that is positive and gets larger as either

1. σ_g^2 increases, so that aggregate risk increases
2. γ increases, so that investors become more risk averse

Qualitatively, these implications seem right on target. The question is whether quantitatively, the model can match the US data.

Testing the CCAPM

To answer this question, Mehra and Prescott use US data from 1889 to 1978 to estimate the mean and standard deviaiton of the log of the gross rate of consumption growth

$$\mu_g = 0.0183 \text{ and } \sigma_g = 0.0357$$

and the mean real (inflation-adjusted) returns on risk-free securities and the Standard & Poor's Composite Stock Price Index

$$r_f = 0.0080 \text{ and } E(r_e) = 0.0698$$

The implied equity risk premium is $E(r_e) - r_f = 0.0618$.

Testing the CCAPM

Consider setting the coefficient of relative risk aversion equal to $\gamma = 2$ and the discount factor equal to $\beta = 0.95$. With $\mu_g = 0.0183$ and $\sigma_g = 0.0357$, the CCAPM implies

$$r_{f,t+1} = \left(\frac{1}{\beta}\right) \exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right) - 1 = 0.0891$$

compared to $r_f = 0.0080$ in the data and

$$E_t(r_{e,t+1}) - r_{f,t+1} = (1 + r_{f,t+1})[\exp\left(\gamma\sigma_g^2\right) - 1] = 0.0028$$

compared to $E(r_e) - r_f = 0.0618$ in the data. The risk-free interest rate is more than 10 times too large and the equity risk premium is more than 20 times too small.

Testing the CCAPM

Alternatively, with $\mu_g = 0.0183$ and $\sigma_g = 0.0357$, consider choosing γ and β to match the two statistics:

$$r_{f,t+1} = \left(\frac{1}{\beta}\right) \exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right) - 1 = 0.0080$$

and

$$E_t(r_{e,t+1}) - r_{f,t+1} = (1 + r_{f,t+1})[\exp\left(\gamma\sigma_g^2\right) - 1] = 0.0618$$

Testing the CCAPM

Since the CCAPM implies that the equity risk premium depends on γ and σ_g^2

$$E_t(r_{e,t+1}) - r_{f,t+1} = (1 + r_{f,t+1})[\exp\left(\gamma\sigma_g^2\right) - 1] = 0.0618$$

can be solved for γ:

$$\gamma = \left(\frac{1}{\sigma_g^2}\right) \ln\left(\frac{0.0618}{1 + r_{f,t+1}} + 1\right)$$

or, with $\sigma_g = 0.0357$ and $r_{f,t+1} = 0.0080$,

$$\gamma = 46.7$$

Testing the CCAPM

And with $\gamma = 46.7$,

$$r_{f,t+1} = \left(\frac{1}{\beta}\right) \exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right) - 1 = 0.0080$$

can be solved for

$$\beta = \left(\frac{1}{1.0080}\right) \exp\left(\gamma\mu_g - \frac{1}{2}\gamma^2\sigma_g^2\right)$$

or, with $\mu_g = 0.0183$ and $\sigma_g = 0.0357$,

$$\beta = 0.58$$

Testing the CCAPM

Thus, the CCAPM can match both the average risk-free rate and the equity risk premium with $\gamma = 46.7$ and $\beta = 0.58$.

To see the problem with setting $\gamma = 46.7$, recall that the certainty equivalent $CE(\tilde{Z})$ for an asset with random payoff $\tilde{Z}$ is the maximum riskless payoff that a risk-averse investor is willing to exchange for that asset. Mathematically,

$$E[u(Y + \tilde{Z})] = u[Y + CE(\tilde{Z})]$$

where Y is the investor's income level without the asset.

Testing the CCAPM

Previously, we calculated the certainty equivalent for an asset that pays 50000 with probability 1/2 and 0 with probability 1/2 when income is 50000 and the coefficient of relative risk aversion is γ.

γ	$CE(\tilde{Z})$
0	25000 (risk neutrality)
1	20711 (logarithmic utility, proposed by D Bernoulli)
2	16667
3	13246
4	10571
5	8566
10	3991
20	1858
50	712

Testing the CCAPM

In particular, an investor with income of 50000 and $\gamma = 46.7$ would take only 764 in exchange for a 50-50 chance of winning another 50000 versus getting/losing nothing.

Intuitively, the CCAPM implies that

$$1 + E_t(r_{e,t+1}) = (1 + r_{f,t+1}) \exp\left(\gamma \sigma_g^2\right)$$

The variance of log consumption growth is small ($\sigma_g = 0.0357$ implies $\sigma_g^2 = 0.0013$), so the model can only account for an equity risk premium of 0.0618 if investors are extremely risk averse.

Testing the CCAPM

Recall, as well, that a risk-averse investor increases his or her saving when asset returns become more volatile if his or her coefficient of relative prudence

$$P_R(Y) = -\frac{Uu'''(Y)}{u''(Y)}$$

exceeds 2.

Previously, we saw that for an investor with CRRA utility, the coefficient of relative prudence equals the coefficient of relative risk aversion plus one: $\gamma + 1$.

Testing the CCAPM

Hence, with $\gamma = 46.7$, investors are not only highly risk averse but also highly prudent. The CCAPM then implies that with $\gamma = 46.7$, investors have a strong motive for allocating savings to the riskless asset. In equilibrium, this strong demand for the riskless asset puts downward pressure on the risk-free rate $r_{f,t+1}$.

The very small value $\beta = 0.58$ is needed to match the average risk-free rate in the data: with higher values of β, the risk-free rate would be too low. But $\beta = 0.58$ implies that in a world of certainty, investors would discount the future by 42 percent per year!

Testing the CCAPM

Thus, while the CCAPM very usefully highlights the qualitative links between aggregate risk and declines in consumption that take place during a recession, Mehra and Prescott's equity premium puzzle is that, for reasonable levels of risk aversion, the CCAPM cannot explain, quantitatively, the size of the equity risk premium observed historically in the US.

Mehra and Prescott's findings have led to an enormous amount of subsequent research asking if there are any modifications to Lucas' original model that can do a better job of matching the data.

Testing the CCAPM

Philippe Weil, "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," *Journal of Monetary Economics* Vol.24 (November 1989): pp.401-421.

Weil asks whether the CCAPM's quantitative problems can be resolved if the vN-M preference specification with CRRA utility is replaced by Epstein and Zin's nonexpected utility function, which allows the coefficient of relative risk aversion to be different from the elasticity of intertemporal substitution.

Testing the CCAPM

Weil finds that with Epstein-Zin preferences, an unrealistically large coefficient of relative risk aversion is still needed to explain the equity risk premium.

But, in addition, with reasonable values for the elasticity of intertemporal substitution, the model again implies that the risk-free rate is much higher than it is in the US data.

Testing the CCAPM

Hence, the added flexibility of the Epstein-Zin preference specification works to underscore that the CCAPM suffers from a risk-free rate puzzle as well as an equity premium puzzle.

The model has great difficulty explaining why the risk-free rate in the US is low as well as why the equity premium is so large.

Testing the CCAPM

Thomas A. Rietz, "The Equity Risk Premium: A Solution," *Journal of Monetary Economics* Vol.22 (July 1988): pp.117-131.

Rietz argues that Mehra and Prescott's estimate of σ_g greatly understates the true amount of aggregate risk in the US economy, if there is a very small chance of an economic disaster and stock market crash that is even worse that what the US experienced during the Great Depression.

Testing the CCAPM

Rietz's argument was dismissed, at first, on the grounds that the odds of an economic disaster of the magnitude required are just too small. But the events of 2008 have rekindled interest in this potential explanation of the equity premium puzzle.

In fact, even before the recent financial crisis, a few papers had already started to take Rietz's hypothesis more seriously, including Robert Barro, "Rare Disasters and Asset Markets int he Twentieth Century," *Quarterly Journal of Economics*, Vol.121 (August 1006): 823-866.

Testing the CCAPM

John Campbell and John Cochrane, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy* Vol.107 (April 1999): 205-251.

Campbell and Cochrane argue that investors may be very risk averse if they dislike declines as well as low levels of consumption.

Testing the CCAPM

In particular, Campbell and Cochrane assume that the representative investor has expected utility

$$E\sum_{t=0}^{\infty}\beta^t u(c_t, s_t)$$

where the Bernoulli utility function still takes the CRRA form

$$u(c_t, s_t) = \frac{(c_t - s_t)^{1-\gamma} - 1}{1-\gamma},$$

but depends not on consumption c_t but on consumption relative to a "habit stock" s_t that is a slow moving average of past consumption.

Testing the CCAPM

With

$$u(c_t, s_t) = \frac{(c_t - s_t)^{1-\gamma} - 1}{1 - \gamma}$$

and hence

$$u'(c) = (c - s)^{-\gamma} \text{ and } u''(c) = -\gamma(c - s)^{-\gamma-1},$$

the coefficient of relative risk aversion equals

$$R_A(c) = -\frac{cu''(c)}{u'(c)} = -\frac{-\gamma(c - s)^{-\gamma-1}}{(c - s)^{-\gamma}} = -\frac{\gamma c}{c - s}$$

so that investors become extreme risk averse when today's consumption c threatens to fall below the habit stock s.

Testing the CCAPM

Campbell and Cochrane's utility function also explains:

1. Why consumers really dislike recessions: because they are averse to even small declines in consumption.
2. Why consumers don't seem much happier today than they were generations ago: because even though the level of consumption today is much higher, so is the habit stock.

Testing the CCAPM

One might wonder, however, where this habit stock comes from, or what it really represents.

And it is still true that Campbell and Cochrane's explanation of the equity risk premium must still appeal to high levels of risk aversion.

Testing the CCAPM

Like the CAPM – and perhaps even more so – the CCAPM is an equilibrium theory of asset prices that very usefully links asset returns to measures of aggregate risk and, from there, to the economy as a whole, but also suffers from important empirical shortcomings.

An active and important line of research in financial economics continues to modify and extend the CCAPM to improve its performance.

Testing the CCAPM

In the meantime, another important strand of research focuses instead on developing no-arbitrage theories, which temporarily set aside the goal of linking asset prices to the overall economy but provide quantitative results that are more reliable and immediately applicable.