

ECON 337901

FINANCIAL ECONOMICS

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Approaches to Valuation

One approach to pricing a risky asset payoff or cash flow \tilde{C} is by taking its expected value $E(\tilde{C})$ and then “penalizing” the riskiness by either discounting at a rate that is higher than the risk-free rate

$$P^A = \frac{E(\tilde{C})}{1 + r_f + \psi}$$

or by reducing its value more directly as

$$P^A = \frac{E(\tilde{C}) - \Psi}{1 + r_f}$$

Approaches to Valuation

$$P^A = \frac{E(\tilde{C})}{1 + r_f + \psi} \text{ or } P^A = \frac{E(\tilde{C}) - \Psi}{1 + r_f}$$

In this context, the CAPM and CCAPM are both models of the risk premia ψ and Ψ .

The CAPM associates the risk premia with the correlation between returns on the risky asset and the market portfolio.

The CCAPM associates the risk premia with the correlation between returns on the risk asset and the IMRS.

Approaches to Valuation

An alternative approach is to break the risky payoff or cash flow \tilde{C} down “state-by-state” C^1, C^2, \dots, C^N and then pricing it as a bundle of contingent claims:

$$P^A = \sum_{i=1}^N q^i C^i$$

In this Arrow-Debreu approach, the contingent claims prices can either be inferred from other asset price through no-arbitrage arguments or linked to investors' IMRS through equilibrium analysis.

Approaches to Valuation

Now we will consider a third approach, the **martingale** or **risk-neutral** approach to pricing, developed in the late 1970s by Michael Harrison and David Kreps, “Martingales and Arbitrage in Multiperiod Securities Markets,” *Journal of Economic Theory* Vol.20 (June 1979): pp.381-408.

This approach is used extensively at the frontiers of asset pricing theory, as in Darrell Duffie, *Dynamic Asset Pricing Theory*, Princeton: Princeton University Press, 2001.

Approaches to Valuation

Although the terminology associated with the two approaches differs, the connections between Arrow-Debreu and martingale pricing methods are very strong.

To an extent, therefore, we can learn about martingale methods simply by taking what we already know about Arrow-Debreu pricing and “translating” our previous insights into the new language.

The Setting and the Intuition

Consider, once again, a setting in which there are two dates, $t = 0$ and $t = 1$, and N possible states $i = 1, 2, \dots, N$ at $t = 1$.

Let π_i , $i = 1, 2, \dots, N$, denote the probability of each state i at $t = 1$.

The Setting and the Intuition

Here is what we know about the **objective, physical**, or true **probability measure**:

$$\pi_i > 0 \text{ for all } i = 1, 2, \dots, N$$

A probability cannot be negative, and if $\pi_i = 0$ you can always delete it by shortening the list of possible states.

$$\sum_{i=0}^N \pi_i = 1$$

If you have accounted for all the possible states, their probabilities must sum to one.

The Setting and the Intuition

In this environment, let there be a risk-free security that sells for $p_b^0 = 1$ at $t = 0$ and pays off $p_b^i = 1 + r_f$ in all states $i = 1, 2, \dots, N$ at $t = 1$. Then r_f is the risk-free rate.

Let there also be $j = 1, 2, \dots, M$ **fundamental** risky securities:

$$p_j^0 = \text{price of security } j \text{ at } t = 0$$

$$p_j^i = \text{payoff from (price of) security } j \text{ in state } i \text{ at } t = 1$$

As in A-D no-arbitrage theory, we will make no assumptions about preferences or the distribution of asset returns.

The Setting and the Intuition

Martingale pricing methods attempt to find a **risk-neutral probability measure**, summarized in this environment by a set of risk-neutral probabilities π_i^{RN} for $i = 1, 2, \dots, N$, which can and usually will differ from the true or objective probabilities, but are such that

$$p_j^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_j^i$$

for each fundamental asset $j = 1, 2, \dots, M$.

The Setting and the Intuition

$$p_j^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_j^i$$

That is, the price at $t = 0$ of each fundamental security must equal

1. The expected value of the price or payoff at $t = 1$, but computed using the risk-neutral instead of the true probability measure.
2. Discounted back to $t = 0$ using the risk-free rate.

Hence, martingale methods correct for risk by adjusting probabilities instead of adjusting the risk-free rate.

The Setting and the Intuition

In probability theory, a **martingale** is a **stochastic process**, that is, a sequence of random variables X_t , $t = 0, 1, 2, \dots$, that satisfies

$$X_t = E_t(X_{t+1})$$

so that the expected value at t of X_{t+1} equals X_t .

Equivalently, a martingale satisfies

$$0 = E_t(X_{t+1} - X_t)$$

implying that it is not expected to change between t and $t + 1$.

The Setting and the Intuition

The classic example of a martingale occurs when you keep track of X_t as your accumulated stock of winnings or losings in a fair coin-flip game, where you gain 1 if the coin comes up heads and lose 1 if the coin comes up tails.

Since the coin flip is “fair,” the probability of either outcome is 0.50 and

$$E_t(X_{t+1}) = X_t + 0.50 \times 1 + 0.50 \times (-1) = X_t$$

The Setting and the Intuition

Since

$$p_j^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_j^i$$

can be rewritten as

$$\frac{p_j^0}{(1 + r_f)^0} = \sum_{i=1}^N \pi_i^{RN} \left[\frac{p_j^i}{(1 + r_f)^1} \right] = E_0^{RN} \left[\frac{p_j^i}{(1 + r_f)^1} \right]$$

the “discounted security price process is a martingale under the risk-neutral probability measure.” Hence the name “martingale pricing.”

The Setting and the Intuition

Mathematically, finding the risk-neutral probabilities amounts to collecting the equations

$$p_j^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_j^i \text{ for all } j = 1, 2, \dots, M$$

and

$$\sum_{i=1}^N \pi_i^{RN} = 1$$

and trying to solve this system of equations subject to the “side conditions”

$$\pi_i^{RN} > 0 \text{ for all } i = 1, 2, \dots, N.$$

The Setting and the Intuition

$$p_j^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_j^i \text{ for all } j = 1, 2, \dots, M$$

$$\sum_{i=1}^N \pi_i^{RN} = 1$$

$$\pi_i^{RN} > 0 \text{ for all } i = 1, 2, \dots, N.$$

Hence, the risk-neutral probabilities must price all M fundamental assets . . .

The Setting and the Intuition

$$p_j^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_j^i \text{ for all } j = 1, 2, \dots, M$$

$$\sum_{i=1}^N \pi_i^{RN} = 1$$

$$\pi_i^{RN} > 0 \text{ for all } i = 1, 2, \dots, N$$

... and the risk-neutral probabilities must sum to one and assign positive probability to the same N states identified by the true probability measure.

The Setting and the Intuition

In probability theory, the two sets of requirements

$$\pi_i > 0 \text{ for all } i = 1, 2, \dots, N$$

and

$$\pi_i^{RN} > 0 \text{ for all } i = 1, 2, \dots, N$$

make the objective and risk-neutral probability measures **equivalent**.

In this context, the notion of equivalence is something more akin to continuity than to equality.

The Setting and the Intuition

$$p_j^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_j^i \text{ for all } j = 1, 2, \dots, M$$

$$\sum_{i=1}^N \pi_i^{RN} = 1$$

$$\pi_i^{RN} > 0 \text{ for all } i = 1, 2, \dots, N.$$

Let's start by considering a case in which this system of equations does **not** have a solution.

The Setting and the Intuition

Suppose that two of the fundamental securities have the same price at $t = 0$; for example:

$$p_1^0 = p_2^0$$

But suppose that security $j = 1$ pays off at least as much as $j = 2$ in every state and more than $j = 1$ in at least one state at $t = 1$; for example:

$$p_1^1 > p_2^1$$

and

$$p_1^i = p_2^i \text{ for all } i = 2, 3, \dots, N$$

The Setting and the Intuition

With

$$p_1^0 = p_2^0 \text{ and } p_1^1 > p_2^1 \text{ and } p_1^i = p_2^i \text{ for all } i = 2, 3, \dots, N$$

the two martingale pricing equations

$$p_1^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_1^i$$

$$p_2^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} p_2^i$$

must have the same left-hand sides, but there is no way to choose $\pi_1^{RN} > 0$ to make this happen.

The Setting and the Intuition

On the other hand, with

$$p_1^0 = p_2^0 \text{ and } p_1^1 > p_2^1 \text{ and } p_1^i = p_2^i \text{ for all } i = 2, 3, \dots, N$$

there is an **arbitrage opportunity**.

The portfolio constructed by buying one share of asset 1 and selling short one share of asset 2 **costs nothing** at $t = 0$ but generates a positive payoff in state 1 at $t = 1$!

The Setting and the Intuition

This is, in fact, one of the main messages of martingale pricing theory.

There is a tight link between the absence of arbitrage opportunities and the existence of a risk-neutral probability measure for pricing securities.

This result is referred to as the **Fundamental Theorem of Asset Pricing** by Philip Dybvig and Stephen Ross, “Arbitrage,” in *The New Palgrave Dictionary of Economics*, 2008.

Definitions and Basic Results

Still working in the two-date, N -state framework, consider a portfolio W consisting of w_b bonds (units of the risk-free asset) and w_j shares (units) of each fundamental risky asset $j = 1, 2, \dots, M$.

As usual, negative and/or fractional values for w_b and the w_j 's are allowed. Assets are “perfectly divisible” and short selling is permitted.

Definitions and Basic Results

The value (cost) of the portfolio at $t = 0$ is

$$V_w^0 = w_b + \sum_{j=1}^M w_j p_j^0$$

and the value (payoff) in each state $i = 1, 2, \dots, N$ at $t = 1$ is

$$V_w^i = w_b(1 + r_f) + \sum_{j=1}^M w_j p_j^i$$

Definitions and Basic Results

The portfolio W constitutes an **arbitrage opportunity** if **all** of the following conditions hold:

1. $V_w^0 = 0$
2. $V_w^i \geq 0$ for all $i = 1, 2, \dots, N$
3. $V_w^i > 0$ for at least one $i = 1, 2, \dots, N$

This definition of an arbitrage opportunity is slightly more specific than the one we've been using up until now. It **requires** no money down today but **allows** for only the **possibility** of profit – with no possibility of loss – in the future.

Definitions and Basic Results

The portfolio W constitutes an **arbitrage opportunity** if **all** of the following conditions hold:

1. $V_w^0 = 0$
2. $V_w^i \geq 0$ for all $i = 1, 2, \dots, N$
3. $V_w^i > 0$ for at least one $i = 1, 2, \dots, N$

The “no money down” requirement (1) is often described by saying that the portfolio must be **self-financing**. Short positions in the portfolio must offset long positions so that, on net, the entire portfolio can be assembled at zero cost.

Definitions and Basic Results

Next, we'll consider three examples:

1. With complete markets and no arbitrage opportunities . . .
2. With incomplete markets and no arbitrage opportunities
...
3. With arbitrage opportunities . . .

And see that

1. . . a unique risk-neutral probability measure exists.
2. . . multiple risk-neutral probability measures exist.
3. . . no risk-neutral probability measure exists.

Definitions and Basic Results

Each example has two periods, $t = 0$ and $t = 1$, and three states $i = 1, 2, 3$, at $t = 1$.

Example 1, with complete markets and no arbitrage opportunities, features the bond and two risky stocks as fundamental securities.

Definitions and Basic Results

Example 1	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	3	2	1
Stock $j = 2$	3	1	4	6

With three assets having linearly independent payoffs, markets are complete; and there are no arbitrage opportunities.

In this setting, a risk-neutral probability measure, if it exists, amounts to choice of π_1^{RN} , π_2^{RN} , and π_3^{RN} .

Definitions and Basic Results

Example 1	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	3	2	1
Stock $j = 2$	3	1	4	6

The risk-neutral probabilities must price the two risky assets

$$2 = \frac{1}{1.1} (3\pi_1^{RN} + 2\pi_2^{RN} + \pi_3^{RN})$$

$$3 = \frac{1}{1.1} (\pi_1^{RN} + 4\pi_2^{RN} + 6\pi_3^{RN})$$

sum to one

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

and satisfy $\pi_1^{RN} > 0$, $\pi_2^{RN} > 0$, and $\pi_3^{RN} > 0$.

Definitions and Basic Results

$$2 = \frac{1}{1.1} (3\pi_1^{RN} + 2\pi_2^{RN} + \pi_3^{RN})$$

$$3 = \frac{1}{1.1} (\pi_1^{RN} + 4\pi_2^{RN} + 6\pi_3^{RN})$$

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

We have a system of 3 **linear** equations in 3 unknowns. The unique solution

$$\pi_1^{RN} = 0.3 \text{ and } \pi_2^{RN} = 0.6 \text{ and } \pi_3^{RN} = 0.1$$

also satisfies the side conditions $\pi_1^{RN} > 0$, $\pi_2^{RN} > 0$, and $\pi_3^{RN} > 0$.

Definitions and Basic Results

Our first example confirms that when markets are complete and there is no arbitrage, there **exists** a risk-neutral probability measure and, moreover, that risk-neutral probability measure is unique.

To see what happens when markets are incomplete, let's drop the second risky stock from our first example. Since there will then be fewer fundamental assets M than states N at $t = 1$, markets will be incomplete, although there will still be no arbitrage opportunities.

Definitions and Basic Results

Example 2	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	3	2	1

The risk-neutral probabilities must price the one risky asset

$$2 = \frac{1}{1.1} (3\pi_1^{RN} + 2\pi_2^{RN} + \pi_3^{RN})$$

sum to one

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

and satisfy $\pi_1^{RN} > 0$, $\pi_2^{RN} > 0$, and $\pi_3^{RN} > 0$.

Definitions and Basic Results

$$2 = \frac{1}{1.1} (3\pi_1^{RN} + 2\pi_2^{RN} + \pi_3^{RN})$$

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

We now have a system of 2 equations in 3 unknowns, albeit one that also imposes the side conditions $\pi_1^{RN} > 0$, $\pi_2^{RN} > 0$, and $\pi_3^{RN} > 0$.

Definitions and Basic Results

$$2 = \frac{1}{1.1} (3\pi_1^{RN} + 2\pi_2^{RN} + \pi_3^{RN})$$

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

To see what possibilities are allowed for, let's temporarily hold π_1^{RN} fixed and use the two equations to solve for

$$\pi_2^{RN} = 1.2 - 2\pi_1^{RN}$$

$$\pi_3^{RN} = \pi_1^{RN} - 0.2$$

Definitions and Basic Results

$$\pi_2^{RN} = 1.2 - 2\pi_1^{RN}$$

$$\pi_3^{RN} = \pi_1^{RN} - 0.2$$

Now see what the side conditions require:

$$\pi_2^{RN} = 1.2 - 2\pi_1^{RN} > 0 \text{ requires } 0.6 > \pi_1^{RN}$$

$$\pi_3^{RN} = \pi_1^{RN} - 0.2 > 0 \text{ requires } \pi_1^{RN} > 0.2$$

Definitions and Basic Results

$$2 = \frac{1}{1.1} (3\pi_1^{RN} + 2\pi_2^{RN} + \pi_3^{RN})$$

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

Evidently, **any** risk-neutral probability measure with

$$0.6 > \pi_1^{RN} > 0.2$$

$$\pi_2^{RN} = 1.2 - 2\pi_1^{RN}$$

$$\pi_3^{RN} = \pi_1^{RN} - 0.2$$

will work.

Definitions and Basic Results

$$2 = \frac{1}{1.1} (3\pi_1^{RN} + 2\pi_2^{RN} + \pi_3^{RN})$$

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

$$0.6 > \pi_1^{RN} > 0.2$$

$$\pi_2^{RN} = 1.2 - 2\pi_1^{RN}$$

$$\pi_3^{RN} = \pi_1^{RN} - 0.2$$

Notice that **all** of the possible risk-neutral probability measures imply the **same** prices for the fundamental securities, which are already being traded. They may, however, imply **different** prices for securities that are not yet traded in this setting with incomplete markets.

Definitions and Basic Results

Hence, our first two examples confirm that it is the absence of arbitrage opportunities that is crucial for the **existence** of a risk-neutral probability measure.

The completeness or incompleteness of markets then determines whether or not the risk-neutral probability measure is **unique**.

As a third example, let's confirm that the presence of arbitrage opportunities, a risk-neutral probability measure fails to exist.

Definitions and Basic Results

Example 3	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	1	2	3
Stock $j = 2$	3	3	4	5

Here, there is an arbitrage opportunity, since buying one share of stock $j = 2$ and selling one bond and one share of stock $j = 1$ costs nothing, on net, at $t = 0$ but generates positive payoffs in all three states at $t = 1$.

Definitions and Basic Results

Example 3	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	1	2	3
Stock $j = 2$	3	3	4	5

The risk-neutral probabilities, if they exist, must price the two risky assets

$$2 = \frac{1}{1.1} (\pi_1^{RN} + 2\pi_2^{RN} + 3\pi_3^{RN})$$

$$3 = \frac{1}{1.1} (3\pi_1^{RN} + 4\pi_2^{RN} + 5\pi_3^{RN})$$

sum to one

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

and satisfy $\pi_1^{RN} > 0$, $\pi_2^{RN} > 0$, and $\pi_3^{RN} > 0$.

Definitions and Basic Results

$$2 = \frac{1}{1.1} (\pi_1^{RN} + 2\pi_2^{RN} + 3\pi_3^{RN})$$

$$3 = \frac{1}{1.1} (3\pi_1^{RN} + 4\pi_2^{RN} + 5\pi_3^{RN})$$

$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

We still have a system of three equations in the three unknowns. If, however, we add the first and third of these equations to get

$$3 = \frac{1}{1.1} (2.1\pi_1^{RN} + 3.1\pi_2^{RN} + 4.1\pi_3^{RN})$$

Definitions and Basic Results

But

$$3 = \frac{1}{1.1} (3\pi_1^{RN} + 4\pi_2^{RN} + 5\pi_3^{RN})$$

$$3 = \frac{1}{1.1} (2.1\pi_1^{RN} + 3.1\pi_2^{RN} + 4.1\pi_3^{RN})$$

cannot both hold, if the probabilities must satisfy the side conditions $\pi_1^{RN} > 0$, $\pi_2^{RN} > 0$, and $\pi_3^{RN} > 0$.

Again, we see that a risk-neutral probability measure fails to exist in the presence of arbitrage opportunities.

Definitions and Basic Results

The lessons from these three examples generalize to yield the following propositions.

Proposition 1 (Fundamental Theorem of Asset Pricing) There exists a risk-neutral probability measure if and only if there are no arbitrage opportunities among the set of fundamental securities.

Definitions and Basic Results

Proposition 2 If there are no arbitrage opportunities among the set of fundamental securities, then the value at $t = 0$ of any portfolio W of the fundamental securities must equal the discounted expected value of the payoffs generated by that portfolio at $t = 1$, when the expected value is computed using **any** risk-neutral probability measure and the discounting uses the risk-free rate:

$$V_w^0 = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} V_w^i$$

Definitions and Basic Results

Proposition 3 Suppose there are no arbitrage opportunities among the set of fundamental securities. Then markets are complete if and only if there exists a **unique** risk-neutral probability measure.

Relation to Arrow-Debreu

Obscured behind differences in terminology are very close links between martingale pricing methods and Arrow-Debreu theory.

To see these links, let's remain in the martingale pricing environment with two periods, $t = 0$ and $t = 1$, and states $i = 1, 2, \dots, N$ at $t = 1$, but imagine as well that the fundamental securities are, in fact, Arrow-Debreu contingent claims.

Relation to Arrow-Debreu

Equivalently, we can assume that markets are complete, so that A-D contingent claims for each state can be constructed as portfolios of the the fundamental securities.

Proposition 2 then implies that these claims can be priced using the risk-neutral probability measure.

Relation to Arrow-Debreu

In either case, the risk-neutral probability measure allows contingent claims to be priced as

$$q^i = \frac{1}{1 + r_f} \pi_i^{RN}$$

for all $i = 1, 2, \dots, N$, since the contingent claim for state i pays off one in that state and zero otherwise.

Hence, if we know the risk-neutral probability measure, we also know all contingent claims prices.

Relation to Arrow-Debreu

Next, sum

$$q^i = \frac{1}{1 + r_f} \pi_i^{RN}$$

over all $i = 1, 2, \dots, N$ to obtain

$$\sum_{i=1}^N q^i = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} = \frac{1}{1 + r_f}$$

By itself, this equation states a no-arbitrage argument: since the payoff from a bond can be replicated by buying a portfolio consisting of $1 + r_f$ contingent claims for each state $i = 1, 2, \dots, N$, the price of this portfolio must equal the bond price $p_b^0 = 1$.

Relation to Arrow-Debreu

But

$$\sum_{i=1}^N q^i = \frac{1}{1+r_f} \text{ and } q^i = \frac{1}{1+r_f} \pi_i^{RN}$$

can also be combined to yield

$$\pi_i^{RN} = \frac{q^i}{\sum_{i=1}^N q^i} \text{ for all } i = 1, 2, \dots, N.$$

Hence, if we know the contingent claims prices, we also know the risk-neutral probability measure.

Relation to Arrow-Debreu

Thus far, we have only relied on the **no-arbitrage** version of A-D theory. But we can deepen our intuition if we are willing to invoke the **equilibrium** conditions

$$q^i = \frac{\beta \pi_i u'(c^i)}{u'(c^0)} \text{ for all } i = 1, 2, \dots, N.$$

where $u(c)$ is a representative investor's Bernoulli utility function, c^0 and c^i denote his or her consumption at $t = 0$ and in state i at $t = 1$, β is the investor's discount factor, and π_i is what we are now calling the **objective** or true probability of state i .

Relation to Arrow-Debreu

Combine

$$q^i = \frac{1}{1 + r_f} \pi_i^{RN}$$

and

$$q^i = \frac{\beta \pi_i u'(c^i)}{u'(c^0)} \text{ for all } i = 1, 2, \dots, N.$$

to obtain

$$\pi_i^{RN} = \beta(1 + r_f) \left[\frac{u'(c^i)}{u'(c^0)} \right] \pi_i$$

Relation to Arrow-Debreu

$$\pi_i^{RN} = \beta(1 + r_f) \left[\frac{u'(c^i)}{u'(c^0)} \right] \pi_i$$

Thus, apart from the **scaling factor** $\beta(1 + r_f)$ that does not depend on the particular state i , the risk-neutral probability π_i^{RN} “twists” the true or objective probability π_i by adding weight if $u'(c^i)$ is large and down-weighting if $u'(c^i)$ is small.

But if the representative investor is risk averse, so that $u(c)$ is concave, when is $u'(c)$ large?

Relation to Arrow-Debreu

$$\pi_i^{RN} = \beta(1 + r_f) \left[\frac{u'(c^i)}{u'(c^0)} \right] \pi_i$$

Thus, apart from the **scaling factor** $\beta(1 + r_f)$ that does not depend on the particular state i , the risk-neutral probability π_i^{RN} “twists” the true or objective probability π_i by adding weight if $u'(c^i)$ is large and down-weighting if $u'(c^i)$ is small.

$u'(c)$ is large during a recession and small during a boom.

Relation to Arrow-Debreu

$$\pi_i^{RN} = \beta(1 + r_f) \left[\frac{u'(c^i)}{u'(c^0)} \right] \pi_i$$

Thus, the risk-neutral probabilities might more accurately be called “risk-adjusted” probabilities, since they “correct” the true probabilities to overweight states in which aggregate outcomes are particularly bad.

This is why martingale pricing reliably discounts risky payoffs and cash flows.

Relation to Arrow-Debreu

$$\pi_i^{RN} = \beta(1 + r_f) \left[\frac{u'(c^i)}{u'(c^0)} \right] \pi_i$$

But like the **no-arbitrage** version of A-D theory, martingale pricing infers the risk-neutral probabilities from observed asset prices, not based on assumptions about preferences and consumption.

Relation to Arrow-Debreu

$$\pi_i^{RN} = \beta(1 + r_f) \left[\frac{u'(c^i)}{u'(c^0)} \right] \pi_i$$

Suppose, however, that the representative investor is **risk-neutral**, with

$$u(c) = a + bc$$

for fixed values of a and $b > 0$, so that $u''(c) = 0$ instead of $u''(c) < 0$.

Relation to Arrow-Debreu

With risk-neutral investors,

$$u(c) = a + bc$$

implies $u'(c) = b$, and

$$\pi_i^{RN} = \beta(1 + r_f) \left[\frac{u'(c^i)}{u'(c^0)} \right] \pi_i$$

collapses to

$$\pi_i^{RN} = \beta(1 + r_f)\pi_i$$

Relation to Arrow-Debreu

With risk-neutral investors

$$\pi_i^{RN} = \beta(1 + r_f)\pi_i$$

the risk-neutral and objective probabilities differ only by a constant scaling factor.

Hence, the term “risk-neutral” probabilities: the risk-neutral probabilities are (up to a constant scaling factor) what probabilities would have to be in an economy where asset prices are the same as what we observe but investors were risk neutral.

Relation to Arrow-Debreu

With risk-neutral investors

$$\pi_i^{RN} = \beta(1 + r_f)\pi_i$$

the risk-neutral and objective probabilities differ only by a constant scaling factor.

Or, put differently, using risk-neutral probabilities instead of objective probabilities allows us to price assets “as if” investors were risk neutral.

Relation to Arrow-Debreu

Finally, consider any asset that yields payoffs X^i in each state $i = 1, 2, \dots, N$ at $t = 1$.

The martingale approach will price this asset using the risk-neutral probabilities as

$$p^A = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} X^i = \frac{1}{1 + r_f} E^{RN}(X^i)$$

where the RN superscript indicates that the expectation is computed using the risk-neutral probabilities.

Relation to Arrow-Debreu

Since, in an Arrow-Debreu equilibrium, the risk-neutral probabilities are linked to the representative investor's IMRS and the objective probabilities as

$$\pi_i^{RN} = (1 + r_f) \left[\frac{\beta u'(c^i)}{u'(c^0)} \right] \pi_i$$

we can rewrite the martingale pricing equation

$$p^A = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} X^i = \frac{1}{1 + r_f} E^{RN}(X^i)$$

in terms of the IMRS and the objective probabilities.

Relation to Arrow-Debreu

$$\begin{aligned} p^A &= \frac{1}{1+r_f} E^{RN}(X^i) = \frac{1}{1+r_f} \sum_{i=1}^N \pi_i^{RN} X^i \\ &= \frac{1}{1+r_f} \sum_{i=1}^N \left\{ (1+r_f) \left[\frac{\beta u'(c^i)}{u'(c^0)} \right] \pi_i \right\} X^i \\ &= \sum_{i=1}^N \pi_i \left[\frac{\beta u'(c^i)}{u'(c^0)} \right] X^i = E \left\{ \left[\frac{\beta u'(c^i)}{u'(c^0)} \right] X^i \right\} \end{aligned}$$

where now the expectation is computed with the **objective probabilities**.

Relation to Arrow-Debreu

Hence, the martingale pricing equation

$$p^A = \frac{1}{1 + r_f} \sum_{i=1}^N \pi_i^{RN} X^i = \frac{1}{1 + r_f} E^{RN}(X^i)$$

is closely linked to the Euler equation from the equilibrium version of Arrow-Debreu theory

$$p^A = \sum_{i=1}^N \pi_i \left[\frac{\beta u'(c^i)}{u'(c^0)} \right] X^i = E \left\{ \left[\frac{\beta u'(c^i)}{u'(c^0)} \right] X^i \right\}$$

Relation to Arrow-Debreu

Let

$$m^i = \frac{\beta u'(c^i)}{u'(c^0)}$$

denote the representative investor's IMRS, so that the A-D Euler equation can be written more compactly as

$$p^A = \sum_{i=1}^N \pi_i \left[\frac{\beta u'(c^i)}{u'(c^0)} \right] X^i = \sum_{i=1}^N m^i X^i = E(m^i X^i)$$

Relation to Arrow-Debreu

In this more compact form

$$p^A = E(m^i X^i)$$

compares more directly to the martingale pricing equation

$$p^A = \frac{1}{1 + r_f} E^{RN}(X^i)$$

In analyses that use the martingale approach, the IMRS m^i is referred to synonymously as the **stochastic discount factor** or the **pricing kernel**.

Relation to Arrow-Debreu

$$p^A = \frac{1}{1 + r_f} E^{RN}(X^i) \text{ and } p^A = E(m^i X^i)$$

highlight the similarities between two equivalent approaches pricing.

1. Deflate by the risk-free rate **after** computing the expectation of the random payoff using the **risk-neutral** probabilities.
2. Deflate with the **stochastic discount factor** **before** computing the expectation of the random payoff using the **objective** probabilities.

Relation to Arrow-Debreu

$$q^i = \frac{1}{1 + r_f} \pi_i^{RN}$$
$$\pi_i^{RN} = \frac{q^i}{\sum_{i=1}^N q^i} \text{ for all } i = 1, 2, \dots, N.$$

Ultimately, the “one-to-one” correspondence between risk-neutral probabilities and contingent claims prices implies that there’s nothing we can do with martingale pricing theory that we cannot do with A-D theory instead – and vice-versa.

Relation to Arrow-Debreu

Still, there are certain complex problems in asset valuation – particularly in pricing options and other derivative securities – and portfolio allocation that are easier, computationally, to solve with martingale methods.

Before moving on, therefore, let's go back to one of our previous examples to see how the martingale approach can lead us to interesting and useful results more quickly than the A-D method does.

Market Incompleteness and Arbitrage Bounds

To illustrate the practical usefulness of martingale pricing methods, let's return to the second example, in which we characterized the multiplicity of risk-neutral probability measures that exist under incomplete markets.

Example 2	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	3	2	1

Here, we have three states at $t = 1$ but only two assets, so markets are incomplete.

Market Incompleteness and Arbitrage Bounds

Example 2	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	3	2	1

What we already know is that the bond, the stock, and **all portfolios** of these existing fundamental securities will be priced, accurately and uniquely, by **any** risk-neutral probability measure with

$$0.6 > \pi_1^{RN} > 0.2$$

$$\pi_2^{RN} = 1.2 - 2\pi_1^{RN}$$

$$\pi_3^{RN} = \pi_1^{RN} - 0.2$$

Market Incompleteness and Arbitrage Bounds

Example 2	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	3	2	1

But suppose one of your clients asks if you will sell him or her a contingent claim for state 1. What price should you ask for, assuming you decide to issue this new security?

Since markets are incomplete, there's no way to “synthesize” that contingent claim by constructing a portfolio of the two existing assets: the bond and stock $j = 1$. There is no “pure arbitrage” argument that will give you the “right” price.

Market Incompleteness and Arbitrage Bounds

The martingale approach, however, tells you that a contingent claim for state 1 ought to sell for

$$q^1 = \frac{1}{1 + r_f} \pi_1^{RN}$$

at $t = 0$, so your data, which tell you that $1 + r_f = 1.1$ and your previous result that **every** risk-neutral probability measure must have $0.6 > \pi_1^{RN} > 0.2$ indicate right away that the price will have to satisfy

$$0.5455 = \frac{0.6}{1.1} > q^i > \frac{0.2}{1.1} = 0.1818$$

Market Incompleteness and Arbitrage Bounds

How can we verify that you are not going to get $0.5455 = 0.6/1.1$ or more if you decide to issue (sell) a contingent claim for state 1 to your client?

Let's go back to the data for the existing assets, and see if there is a way that your client can construct a portfolio of the bond and stock that:

1. Costs $0.6/1.1 = 0.5455$ at $t = 0$
2. Pays off one in state 1 at $t = 1$
3. Pays off **at least** zero in states 2 and 3 at $t = 1$.

Market Incompleteness and Arbitrage Bounds

Is there is a way that your client can construct a portfolio of the bond and stock that:

1. Costs $0.6/1.1 = 0.5455$ at $t = 0$
2. Pays off one in state 1 at $t = 1$
3. Pays off **at least** zero in states 2 and 3 at $t = 1$

Such a portfolio will be better for your client than a contingent claim for state 1; if it exists, that will confirm that you will not get 0.5455 (or more) for selling the claim.

Market Incompleteness and Arbitrage Bounds

Example 2	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	3	2	1

A portfolio consisting of w_b bonds and w_1 shares of stock will cost

$$V_w^0 = w_b + 2w_1$$

at $t = 0$ and will have payoffs

$$V_w^1 = 1.1w_b + 3w_1$$

$$V_w^2 = 1.1w_b + 2w_1$$

$$V_w^3 = 1.1w_b + w_1$$

in the three possible states at $t = 1$.

Market Incompleteness and Arbitrage Bounds

$$V_w^0 = w_b + 2w_1$$

$$V_w^1 = 1.1w_b + 3w_1$$

$$V_w^2 = 1.1w_b + 2w_1$$

$$V_w^3 = 1.1w_b + w_1$$

Your client wants

$$V_w^0 = w_b + 2w_1 = \frac{0.6}{1.1}$$

$$V_w^1 = 1.1w_b + 3w_1 = 1$$

Market Incompleteness and Arbitrage Bounds

Your client wants

$$V_w^0 = w_b + 2w_1 = \frac{0.6}{1.1}$$

$$V_w^1 = 1.1w_b + 3w_1 = 1$$

This is a system of two linear equations in two unknowns, with solution

$$w_b = -\frac{1}{2.2} \text{ and } w_s = \frac{1}{2}$$

Market Incompleteness and Arbitrage Bounds

But the choices

$$w_b = -\frac{1}{2.2} \text{ and } w_s = \frac{1}{2}$$

also imply

$$V_w^2 = 1.1w_b + 2w_1 = 0.5 > 0$$

$$V_w^3 = 1.1w_b + w_1 = 0$$

So this portfolio is **better** for your client than a contingent claim for state 1. If you try to sell the claim for 0.5455 or more, he or she will just buy the portfolio of the bond and stock instead.

Market Incompleteness and Arbitrage Bounds

$$0.5455 = \frac{0.6}{1.1} > q^i > \frac{0.2}{1.1} = 0.1818$$

Now let's ask: should you sell the claim for 0.1818 or less?

The answer is no, if there is a way for you to construct a portfolio of the bond and stock that

1. Provides you with $0.2/1.1 = 0.1818$ at $t = 0$
2. Requires you to make a payment of one in state 1 at $t = 0$
3. Provides you with **at least** zero in states 2 and 3 at $t = 1$

Market Incompleteness and Arbitrage Bounds

Example 2	p_j^0	p_j^1	p_j^2	p_j^3
Bond $j = b$	1	1.1	1.1	1.1
Stock $j = 1$	2	3	2	1

A portfolio consisting of w_b bonds and w_1 shares of stock will cost

$$V_w^0 = w_b + 2w_1$$

at $t = 0$ and will have payoffs

$$V_w^1 = 1.1w_b + 3w_1$$

$$V_w^2 = 1.1w_b + 2w_1$$

$$V_w^3 = 1.1w_b + w_1$$

in the three possible states at $t = 1$.

Market Incompleteness and Arbitrage Bounds

$$V_w^0 = w_b + 2w_1$$

$$V_w^1 = 1.1w_b + 3w_1$$

$$V_w^2 = 1.1w_b + 2w_1$$

$$V_w^3 = 1.1w_b + w_1$$

You want

$$V_w^0 = w_b + 2w_1 = -\frac{0.2}{1.1}$$

$$V_w^1 = 1.1w_b + 3w_1 = -1$$

where the numbers are negative because you want to **receive** the negative price at $t = 0$ and **make** the negative payoff at $t = 1$.

Market Incompleteness and Arbitrage Bounds

You want

$$V_w^0 = w_b + 2w_1 = -\frac{0.2}{1.1}$$

$$V_w^1 = 1.1w_b + 3w_1 = -1$$

This is another system of two linear equations in two unknowns, with solution

$$w_b = \frac{2}{1.1} \text{ and } w_s = -1$$

Market Incompleteness and Arbitrage Bounds

But the choices

$$w_b = \frac{2}{1.1} \text{ and } w_s = -1$$

also imply

$$V_w^2 = 1.1w_b + 2w_1 = 0$$

$$V_w^3 = 1.1w_b + w_1 = 1 > 0$$

So buying portfolio is **better** for you than selling the contingent claim. If your client won't pay more than 0.1818 for the claim, you should buy this portfolio (or do nothing) instead.

Market Incompleteness and Arbitrage Bounds

It took us awhile to confirm the result, but that underscores the fact that the martingale approach initially led us to answer very quickly.

Martingale pricing methods require you to learn a new language, and the economic intuition is not as direct as it is with Arrow-Debreu.

But these methods can be extremely useful in solving very difficult problems in asset pricing.