

# ECON 337901

# FINANCIAL ECONOMICS

Peter Ireland

Boston College

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## 9 Arrow-Debreu Pricing: Equilibrium

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## Arrow-Debreu vs CAPM

The Arrow-Debreu framework was developed in the 1950s and 1960s by Kenneth Arrow (US, b.1932, Nobel Prize 1972) and Gerard Debreu (France, 1921-2004, Nobel Prize 1983), some key references being:

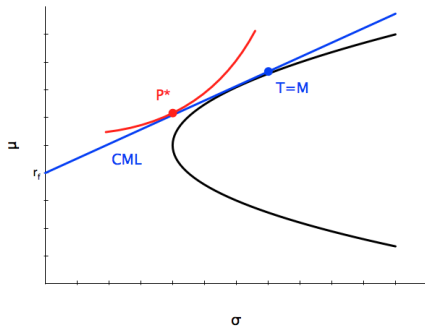
Gerard Debreu, *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, New Haven: Yale University Press, 1959.

Kenneth Arrow, "The Role of Securities in the Optimal Allocation of Risk-Bearing," *Review of Economic Studies* Vol.31 (April 1964): pp.91-96.

## Arrow-Debreu vs CAPM

Since MPT and the CAPM were developed around the same time, it is useful to consider how each of these two approaches brings economic analysis to bear on the problem of pricing risky assets and cash flows.

## Arrow-Debreu vs CAPM



Markowitz, Sharpe, Lintner, and Mossin put  $\sigma^2$  on one axis and  $\mu$  on the other.

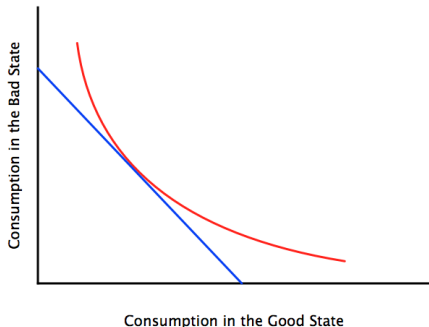
## Arrow-Debreu vs CAPM

Markowitz, Sharpe, Lintner, and Mossin put  $\sigma^2$  on one axis and  $\mu$  on the other.

This allowed them to make more rapid progress, deriving important results for portfolio management and asset pricing.

But the resulting theory proved very difficult to generalize: it requires either quadratic utility or normally distributed returns.

## Arrow-Debreu vs CAPM



Arrow and Debreu put consumption in the good state on one axis and consumption in the bad state on the other.

## Arrow-Debreu vs CAPM

Arrow and Debreu put consumption in the good state on one axis and consumption in the bad state on the other.

Their theory requires none of the restrictive assumptions that underly the CAPM.

But it took much longer to recognize its implications for asset pricing.



## Arrow-Debreu vs CAPM

In the end, both approaches proved to be incredibly valuable.

Markowitz, Sharpe, Arrow, and Debreu all won Nobel Prizes.

## Arrow-Debreu vs CAPM

Advantages of the Arrow-Debreu approach:

1. Investors do not have to have quadratic utility – or even expected utility – functions.
2. Returns do not need to be normally distributed.
3. The theory draws very explicit links between asset prices and the rest of the economy.

# The Arrow-Debreu Economy

Key features of our version of the A-D economy:

1. Two dates:  $t = 0$  (today, when assets are purchased) and  $t = 1$  (the future, when payoffs are received). This can (and will) be generalized.
2.  $N$  possible states at  $t = 1$ , with probabilities  $\pi_i$ ,  $i = 1, 2, \dots, N$ .
3. One perishable (non-storable) good at each date (more goods can be added and the possibility of storage introduced, at the cost of more notational complexity).
4. Individuals initially receive goods as endowments (but production could be introduced, again at the cost of more notational complexity).
5.  $K$  investors,  $j = 1, 2, \dots, K$ , who may differ in their preferences and endowments.

# The Arrow-Debreu Economy

Let

$w_j^0 =$  agent  $j$ 's endowment at  $t = 0$

$w_j^i =$  agent  $j$ 's endowment in state  $i$  at  $t = 1$

$c_j^0 =$  agent  $j$ 's consumption at  $t = 0$

$c_j^i =$  agent  $j$ 's consumption in state  $i$  at  $t = 1$

# The Arrow-Debreu Economy

Use consumption at  $t = 0$  as the “numeraire,” that is, the good in terms of which all other prices are quoted.

Let  $q^i$  be the price at  $t = 0$ , measured in units of  $t = 0$  consumption, of a contingent claim that pays off one unit of consumption in a particular state  $i$  at  $t = 1$  and zero otherwise.

# The Arrow-Debreu Economy

For simplicity, assume that each investor first uses the contingent claims market to sell off his or her endowments at  $t = 0$  and in each state at  $t = 1$ , then uses the same markets to buy back consumption at  $t = 0$  and in each state at  $t = 1$ .

Then we won't need additional notation to keep track of purchases and sales of contingent claims: purchases coincide with consumption and sales with endowments.

## The Arrow-Debreu Economy

Investor  $j$  in our economy therefore faces the budget constraint

$$w_j^0 + \sum_{i=1}^N q^i w_j^i \geq c_j^0 + \sum_{i=1}^N q^i c_j^i$$

Note that we can always go back and compute net sales

$$w_j^0 - c_j^0 \text{ and } w_j^i - c_j^i \text{ for all } i = 1, 2, \dots, N$$

or purchases

$$c_j^0 - w_j^0 \text{ and } c_j^i - w_j^i \text{ for all } i = 1, 2, \dots, N$$

of contingent claims if these turn out to be of interest.

# The Arrow-Debreu Economy

Investors are allowed to have different utility functions, and in the most general A-D model need not even have expected utility functions.

But to obtain sharper results, we will assume that all investors maximize vN-M expected utility functions, but are allowed to have different Bernoulli utility functions reflecting possibly different attitudes towards risk.



## The Arrow-Debreu Economy

Thus, investor  $j$  chooses  $c_j^0$  and  $c_j^i$  for all  $i = 1, 2, \dots, N$  to maximize

$$u_j(c_j^0) + \beta E[u_j(\tilde{c}_j)] = u_j(c_j^0) + \beta \sum_{i=1}^N \pi_i u_j(c_j^i)$$

where the **discount factor**  $\beta$  is again a measure of patience, subject to the budget constraint

$$w_j^0 + \sum_{i=1}^N q^i w_j^i \geq c_j^0 + \sum_{i=1}^N q^i c_j^i$$

# The Arrow-Debreu Economy

Note, again, Arrow and Debreu's key insight: the mathematical structure of this investor's problem is **identical** to the problem faced by a consumer who must divide his or her income up into amounts to be spent on apples, bananas, oranges, etc.

Extending our version of the model to include more than two periods would require even more notation(!). But, both conceptually and mathematically, that extension simply amounts to introducing more goods: pears, pineapples, etc.

# The Arrow-Debreu Economy

In the A-D economy, as in microeconomics more generally, each individual investor operating in perfectly competitive markets takes prices as given and sees him or herself as being able to purchase as much or as little of each good at those competitive prices.

But again as in microeconomics more generally, all markets must clear: for each good, the quantity supplied must equal the quantity demanded.

## The Arrow-Debreu Economy

In our version of the A-D economy, market clearing requires that

$$\sum_{j=1}^K w_j^0 = \sum_{j=1}^K c_j^0$$

and

$$\sum_{j=1}^K w_j^i = \sum_{j=1}^K c_j^i$$

for all  $i = 1, 2, \dots, N$ . These conditions simultaneously describe equilibrium in the market for goods and for contingent claims.

# The Arrow-Debreu Economy

In our A-D economy, therefore, a **competitive equilibrium** consists of a set of consumptions  $c_j^0$  for all  $j = 1, 2, \dots, K$  and  $c_j^i$  for all  $i = 1, 2, \dots, N$  and  $j = 1, 2, \dots, K$  and a set of prices  $q^i$   $i = 1, 2, \dots, N$  such that:

1. Given prices, each investor's consumptions maximize his or her utility subject to his or her budget constraint.
2. All markets clear.

# Competitive Equilibrium and Pareto Optimum

Let's focus first on the first requirement for a CE: investor  $j$  chooses  $c_j^0$  and  $c_j^i$  for all  $i = 1, 2, \dots, N$  to maximize

$$u_j(c_j^0) + \beta \sum_{i=1}^N \pi_i u_j(c_j^i)$$

subject to the budget constraint

$$w_j^0 + \sum_{i=1}^N q^i w_j^i \geq c_j^0 + \sum_{i=1}^N q^i c_j^i$$

# Competitive Equilibrium and Pareto Optimum

The Lagrangian for the investor's problem

$$u_j(c_j^0) + \beta \sum_{i=1}^N \pi_i u_j(c_j^i) + \lambda_j \left( w_j^0 + \sum_{i=1}^N q^i w_j^i - c_j^0 - \sum_{i=1}^N q^i c_j^i \right)$$

leads us to the first-order conditions

$$u_j'(c_j^0) - \lambda_j = 0$$

$$\beta \pi_i u_j'(c_j^i) - \lambda_j q^i = 0 \text{ for all } i = 1, 2, \dots, N$$

## Competitive Equilibrium and Pareto Optimum

$$u'_j(c_j^0) - \lambda_j = 0$$

$$\beta \pi_i u'_j(c_j^i) - \lambda_j q^i = 0 \text{ for all } i = 1, 2, \dots, N$$

imply

$$\beta \pi_i u'_j(c_j^i) = u'_j(c_j^0) q^i \text{ for all } i = 1, 2, \dots, N$$

or

$$q^i = \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \text{ for all } i = 1, 2, \dots, N$$



## Competitive Equilibrium and Pareto Optimum

$$q^i = \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \text{ for all } i = 1, 2, \dots, N$$

The investor, as an **actor** in this economy, takes the price  $q^i$  as given and uses it to choose  $c_j^0$  and  $c_j^i$  optimally.

But we, as **observers** of this economy, can use this optimality condition to see what the investor's choices of  $c_j^0$  and  $c_j^i$  tell us about the contingent claim price  $q^i$  and, by extension, about asset prices more broadly.

## Competitive Equilibrium and Pareto Optimum

$$q^i = \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \text{ for all } i = 1, 2, \dots, N$$

The price  $q^i$  tends to be higher when:

1.  $\beta$  is larger, indicating that investors are more patient.
2.  $\pi_i$  is larger, indicating that state  $i$  is more likely.
3.  $u'_j(c_j^i)$  is larger or  $u'_j(c_j^0)$  is smaller.

## Competitive Equilibrium and Pareto Optimum

$$q^i = \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \text{ for all } i = 1, 2, \dots, N$$

tends to be higher when  $u'_j(c_j^i)$  is larger or  $u'_j(c_j^0)$  is smaller.

If  $u_j$  is concave, that is, if investor  $j$  is risk averse, then a larger value of  $u'_j(c_j^i)$  corresponds to a smaller value of  $c_j^i$  and a smaller value of  $u'_j(c_j^0)$  corresponds to a larger value of  $c_j^0$ .

## Competitive Equilibrium and Pareto Optimum

$$q^i = \frac{\beta \pi_i u_j'(c_j^i)}{u_j'(c_j^0)} \text{ for all } i = 1, 2, \dots, N$$

tends to be higher when  $c_j^i$  is smaller or  $c_j^0$  is larger.

That is,  $q^i$  is higher if investor  $j$ 's consumption falls between  $t = 0$  and state  $i$  at  $t = 1$ .

And this same condition must hold for **all** investors in the economy. Hence,  $q^i$  is higher if everyone expects consumption to fall in state  $i$ .

## Competitive Equilibrium and Pareto Optimum

$$q^i = \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \text{ for all } i = 1, 2, \dots, N$$

When is consumption most likely to fall for **everyone** in the economy? During a recession.

Hence, the A-D model associates a high contingent claim price  $q^i$  with a recession, drawing an explicit link between asset prices and the rest of the economy that is, at best, implicit in the CAPM.

# Competitive Equilibrium and Pareto Optimum

This highlights an important, but somewhat subtle, distinction: the CAPM describes the behavior of **asset returns**, while the A-D model describes the behavior of **asset prices**.

To more directly compare the implications of these two models, let's translate the A-D model's implications for prices into implications for asset returns instead.

## Competitive Equilibrium and Pareto Optimum

In the A-D model, a contingent claim that costs  $q^i$  at  $t = 0$  pays off one unit of consumption in state  $i$  at  $t = 1$ .

Hence, the **return** on this asset between  $t = 0$  and state  $i$  at  $t = 1$  is

$$1 + r^i = \frac{1}{q^i} \text{ or } r^i = \frac{1}{q^i} - 1$$

A **high price**  $q^i$  corresponds to a **low return**  $r^i$ .

# Competitive Equilibrium and Pareto Optimum

Hence, the A-D model associates recessions with low asset returns.

Consistent with CAPM intuition, investors are willing to accept a low return on a contingent claim that pays off during a recession because that asset provides **insurance**.

The difference between low return on this asset and higher returns that may be available on other assets is like the premium that investors are willing to pay for this insurance.



# Competitive Equilibrium and Pareto Optimum

A user of the CAPM would say, “well, recessions are times when the return on the market portfolio is low, so if you want an asset that pays off well during those times, you’ll have to pay a premium.”

The models are fully consistent.

But the A-D model makes the links between asset prices and the rest of the economy more explicit.

# Competitive Equilibrium and Pareto Optimum

To derive some additional implications of the A-D model, let's assume now that all investors have a logarithmic Bernoulli utility function

$$u_j(c) = \ln(c)$$

so that

$$u'_j(c) = \frac{1}{c}$$

for all  $j = 1, 2, \dots, K$ , and that the common discount factor is  $\beta = 0.9$ .

# Competitive Equilibrium and Pareto Optimum

But let's assume, as well, that investors differ in terms of their endowments.

More specifically, suppose that the economy consists of two types of investors – type 1 and type 2 – in equal numbers.

Now we can consider a “representative” of each type:  $j = 1$  and  $j = 2$ .

# Competitive Equilibrium and Pareto Optimum

Suppose for simplicity that there are only two possible states,  $i = 1$  and  $i = 2$ , at  $t = 1$  which occur with probabilities  $\pi_1 = 2/3$  and  $\pi_2 = 1/3$ .

Endowments:

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

# Competitive Equilibrium and Pareto Optimum

In this economy, there are three sets of requirements for a CE:

1. Type 1 investors take prices as given and choose consumptions optimally.
2. Type 2 investors take prices as given and choose consumptions optimally.
3. All markets clear.

Let's consider what each set of requirements implies.

# Competitive Equilibrium and Pareto Optimum

The representative type 1 investor takes  $q^1$  and  $q^2$  as given chooses  $c_1^0$ ,  $c_1^1$ , and  $c_1^2$  to maximize

$$\ln(c_1^0) + 0.9 \left[ (2/3) \ln(c_1^1) + (1/3) \ln(c_1^2) \right]$$

subject to

$$8 + 20q^1 + 13q^2 \geq c_1^0 + q^1 c_1^1 + q^2 c_1^2$$

# Competitive Equilibrium and Pareto Optimum

The Lagrangian

$$L = \ln(c_1^0) + 0.9 \left[ (2/3) \ln(c_1^1) + (1/3) \ln(c_1^2) \right] \\ + \lambda_1 \left[ 8 + 20q^1 + 13q^2 - c_1^0 - q^1 c_1^1 - q^2 c_1^2 \right]$$

leads us to the first-order conditions

$$\frac{1}{c_1^0} = \lambda_1$$

$$0.9(2/3) \left( \frac{1}{c_1^1} \right) = \lambda_1 q^1$$

$$0.9(1/3) \left( \frac{1}{c_1^2} \right) = \lambda_1 q^2$$

## Competitive Equilibrium and Pareto Optimum

$$\frac{1}{c_1^0} = \lambda_1$$

$$0.9(2/3) \left( \frac{1}{c_1^1} \right) = \lambda_1 q^1$$

$$0.9(1/3) \left( \frac{1}{c_1^2} \right) = \lambda_1 q^2$$

Use the first of these three equations to eliminate  $\lambda_1$  in the other two:

$$0.9(2/3) \left( \frac{c_1^0}{c_1^1} \right) = q^1$$

$$0.9(1/3) \left( \frac{c_1^0}{c_1^2} \right) = q^2$$



## Competitive Equilibrium and Pareto Optimum

If we were only interested in solving the individual investor's problem, we could use the two first-order conditions

$$0.9(2/3) \left( \frac{c_1^0}{c_1^1} \right) = q^1$$

$$0.9(1/3) \left( \frac{c_1^0}{c_1^2} \right) = q^2$$

and the budget constraint

$$8 + 20q^1 + 13q^2 = c_1^0 + q^1 c_1^1 + q^2 c_1^2$$

to solve for  $c_1^0$ ,  $c_1^1$ , and  $c_1^2$  in terms of  $q^1$  and  $q^2$ .

# Competitive Equilibrium and Pareto Optimum

The representative type 2 investor takes  $q^1$  and  $q^2$  as given chooses  $c_2^0$ ,  $c_2^1$ , and  $c_2^2$  to maximize

$$\ln(c_2^0) + 0.9 \left[ (2/3) \ln(c_2^1) + (1/3) \ln(c_2^2) \right]$$

subject to

$$12 + 5q^1 + 2q^2 \geq c_2^0 + q^1 c_2^1 + q^2 c_2^2$$

## Competitive Equilibrium and Pareto Optimum

Setting up the Lagrangian, taking the first-order conditions, and eliminating  $\lambda_2$  just as before, we could then use two first-order conditions

$$0.9(2/3) \left( \frac{c_2^0}{c_2^1} \right) = q^1$$

$$0.9(1/3) \left( \frac{c_2^0}{c_2^2} \right) = q^2$$

and the budget constraint

$$12 + 5q^1 + 2q^2 = c_2^0 + q^1 c_2^1 + q^2 c_2^2$$

to solve for  $c_2^0$ ,  $c_2^1$ , and  $c_2^2$  in terms of  $q^1$  and  $q^2$ .

# Competitive Equilibrium and Pareto Optimum

Endowments:

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

Since there are equal numbers of each investor type, market clearing requires

$$w_1^0 + w_2^0 = 20 = c_1^0 + c_2^0$$

$$w_1^1 + w_2^1 = 25 = c_1^1 + c_2^1$$

$$w_1^2 + w_2^2 = 15 = c_1^2 + c_2^2$$

# Competitive Equilibrium and Pareto Optimum

You might at this point be worried. We seem to have 9 equations that must be satisfied: two first-order conditions and one budget constraint for each of the two representative investors and three market-clearing conditions.

But we have only 8 “unknowns” to solve for:  $c_1^0$ ,  $c_1^1$ ,  $c_1^2$ ,  $c_2^0$ ,  $c_2^1$ ,  $c_2^2$ ,  $q^1$ , and  $q^2$

# Competitive Equilibrium and Pareto Optimum

But there is actually no problem: What we have discovered, instead, is a special case of **Walras' Law**, named after Léon Walras (France, 1834-1910).

Walras' Law says that in an economy with  $K$  consumers and  $M$  markets, if

1. All  $K$  consumers' budget constraints are satisfied and
2.  $M - 1$  markets are in equilibrium

then the  $M$ th market must be in equilibrium as well.

# Competitive Equilibrium and Pareto Optimum

Walras' Law implies that we really have only 8 equations in 8 unknowns.

Having the same number of equations as unknowns doesn't guarantee that there will be a solution or that a solution, if it exists, will be unique.

But it does suggest that a unique solution might be found. In fact, you can check that for this economy, all of the equilibrium conditions are satisfied with . . .

# Competitive Equilibrium and Pareto Optimum

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
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## Endowments

type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

## Consumptions

type 1	$c_1^0 = 12$	$c_1^1 = 15$	$c_1^2 = 9$
type 2	$c_2^0 = 8$	$c_2^1 = 10$	$c_2^2 = 6$

## Prices

$q^1 = 0.48$     $q^2 = 0.40$



# Competitive Equilibrium and Pareto Optimum

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
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## Endowments

type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

## Prices

$$q^1 = 0.48 \quad q^2 = 0.40$$

$q^1$  is larger than  $q^2$  because the probability of state 1 is  $2/3$  and the probability of state 2 is  $1/3$  ...

# Competitive Equilibrium and Pareto Optimum

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
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## Endowments

type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

## Prices

$$q^1 = 0.48 \quad q^2 = 0.40$$

... but the difference in prices across states is smaller than the difference in probabilities, because state 1 is a “boom” and state 2 is a “recession.”

# Competitive Equilibrium and Pareto Optimum

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
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## Endowments

type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

## Consumptions

type 1	$c_1^0 = 12$	$c_1^1 = 15$	$c_1^2 = 9$
type 2	$c_2^0 = 8$	$c_2^1 = 10$	$c_2^2 = 6$

Type 1 agents use asset markets to “borrow” against future income and consume more today.

# Competitive Equilibrium and Pareto Optimum

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
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## Endowments

type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

## Consumptions

type 1	$c_1^0 = 12$	$c_1^1 = 15$	$c_1^2 = 9$
type 2	$c_2^0 = 8$	$c_2^1 = 10$	$c_2^2 = 6$

Type 2 agents use asset markets to “lend” and consume in the future.

# Competitive Equilibrium and Pareto Optimum

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
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## Endowments

type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

## Consumptions

type 1	$c_1^0 = 12$	$c_1^1 = 15$	$c_1^2 = 9$
type 2	$c_2^0 = 8$	$c_2^1 = 10$	$c_2^2 = 6$

Both agent types use asset markets to smooth consumption over time and across states.

# Competitive Equilibrium and Pareto Optimum

	$t = 1$	$t = 1$
$t = 0$	state 1	state 2

## Endowments

type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

## Consumptions

type 1	$c_1^0 = 12$	$c_1^1 = 15$	$c_1^2 = 9$
type 2	$c_2^0 = 8$	$c_2^1 = 10$	$c_2^2 = 6$

But neither agent is able to fully insure against the **aggregate risk** of the recession in state 2.

# Competitive Equilibrium and Pareto Optimum

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
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## Endowments

type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$

## Consumptions

type 1	$c_1^0 = 12$	$c_1^1 = 15$	$c_1^2 = 9$
type 2	$c_2^0 = 8$	$c_2^1 = 10$	$c_2^2 = 6$

Type 1 agents are wealthier: their total endowment is worth more and they always consume more.

# Competitive Equilibrium and Pareto Optimum

The A-D economy is one in which the two welfare theorems of economics hold: the resource allocation from a competitive equilibrium is Pareto optimal, and any Pareto optimal resource allocation can be supported in a competitive equilibrium.

To see this, let's continue to work with the special case with only two types of investors and two possible states at  $t = 1$ .



## Competitive Equilibrium and Pareto Optimum

But let's generalize slightly, by assuming that both types of investors have a Bernoulli utility function of the more general form

$$u(c_j^0) + \beta[\pi_1 u(c_j^1) + \pi_2 u(c_j^2)]$$

where  $\pi_1$  is the probability of state 1 and  $\pi_2 = 1 - \pi_1$  is the probability of state 2, and by letting  $w^0$ ,  $w^1$ , and  $w^2$  denote the aggregate endowments at  $t = 0$  and the two states at  $t = 1$ .

## Competitive Equilibrium and Pareto Optimum

Now consider a social planner, who divides the aggregate endowments up into amounts allocated to each of the two representative investors subject to the resource constraints

$$w^0 \geq c_1^0 + c_2^0$$

$$w^1 \geq c_1^1 + c_2^1$$

$$w^2 \geq c_1^2 + c_2^2$$

in order to maximize a weighted sum of their expected utilities

$$\begin{aligned} & \theta \{ u(c_1^0) + \beta[\pi_1 u(c_1^1) + \pi_2 u(c_1^2)] \} \\ & + (1 - \theta) \{ u(c_2^0) + \beta[\pi_1 u(c_2^1) + \pi_2 u(c_2^2)] \} \end{aligned}$$

## Competitive Equilibrium and Pareto Optimum

The Lagrangian for this social planner's problem

$$\begin{aligned} L = & \theta \{ u(c_1^0) + \beta[\pi_1 u(c_1^1) + \pi_2 u(c_1^2)] \} \\ & + (1 - \theta) \{ u(c_2^0) + \beta[\pi_1 u(c_2^1) + \pi_2 u(c_2^2)] \} \\ & + \lambda^0 (w^0 - c_1^0 - c_2^0) + \lambda^1 (w^1 - c_1^1 - c_2^1) \\ & + \lambda^2 (w^2 - c_1^2 - c_2^2) \end{aligned}$$

leads to the first-order conditions

$$\theta u'(c_1^0) = \lambda^0 \text{ and } (1 - \theta) u'(c_2^0) = \lambda^0$$

$$\theta \beta \pi_1 u'(c_1^1) = \lambda^1 \text{ and } (1 - \theta) \beta \pi_1 u'(c_2^1) = \lambda^1$$

$$\theta \beta \pi_2 u'(c_1^2) = \lambda^2 \text{ and } (1 - \theta) \beta \pi_2 u'(c_2^2) = \lambda^2$$

## Competitive Equilibrium and Pareto Optimum

$$\theta u'(c_1^0) = \lambda^0 \text{ and } (1 - \theta)u'(c_2^0) = \lambda^0$$

$$\theta\beta\pi_1 u'(c_1^1) = \lambda^1 \text{ and } (1 - \theta)\beta\pi_1 u'(c_2^1) = \lambda^1$$

$$\theta\beta\pi_2 u'(c_1^2) = \lambda^2 \text{ and } (1 - \theta)\beta\pi_2 u'(c_2^2) = \lambda^2$$

can be combined to obtain

$$\frac{\beta\pi_1 u'(c_1^1)}{u'(c_1^0)} = \frac{\beta\pi_1 u'(c_2^1)}{u'(c_2^0)}$$

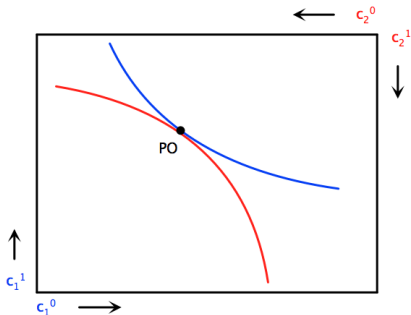
$$\frac{\beta\pi_2 u'(c_1^2)}{u'(c_1^0)} = \frac{\beta\pi_2 u'(c_2^2)}{u'(c_2^0)}$$

## Competitive Equilibrium and Pareto Optimum

$$\frac{\beta\pi_1 u'(c_1^1)}{u'(c_1^0)} = \frac{\beta\pi_1 u'(c_2^1)}{u'(c_2^0)}$$
$$\frac{\beta\pi_2 u'(c_1^2)}{u'(c_1^0)} = \frac{\beta\pi_2 u'(c_2^2)}{u'(c_2^0)}$$

The social planner equates marginal rates of substitution between  $t = 0$  and each state at  $t = 1$  across the two agent types.

# Competitive Equilibrium and Pareto Optimum



The social planner equates marginal rates of substitution between  $t = 0$  and each state at  $t = 1$  across the two agent types.

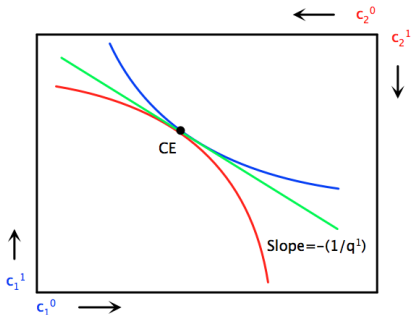
# Competitive Equilibrium and Pareto Optimum

In any competitive equilibrium, however, **both** agent types will equate the same marginal rates of substitution to the contingent claim prices:

$$\frac{\beta\pi_1 u'(c_1^1)}{u'(c_1^0)} = q^1 = \frac{\beta\pi_1 u'(c_2^1)}{u'(c_2^0)}$$

$$\frac{\beta\pi_2 u'(c_1^2)}{u'(c_1^0)} = q^2 = \frac{\beta\pi_2 u'(c_2^2)}{u'(c_2^0)}$$

# Competitive Equilibrium and Pareto Optimum



In any competitive equilibrium, however, **both** agent types will equate the same marginal rates of substitution to the contingent claim prices.



# Competitive Equilibrium and Pareto Optimum

Hence, in the A-D economy as in the Edgeworth box:

**First Welfare Theorem of Economics** The resource allocation from a competitive equilibrium is Pareto optimal.

**Second Welfare Theorem of Economics** A Pareto optimal resource allocation can be supported in a competitive equilibrium.

## Competitive Equilibrium and Pareto Optimum

In our numerical example with log utility, optimal allocations must satisfy

$$\frac{\theta}{c_1^0} = \frac{1 - \theta}{c_2^0} \text{ and } 20 = c_1^0 + c_2^0$$

$$\frac{0.9\theta(2/3)}{c_1^1} = \frac{0.9(1 - \theta)(2/3)}{c_2^1} \text{ and } 25 = c_1^1 + c_2^1$$

$$\frac{0.9\theta(1/3)}{c_1^2} = \frac{0.9(1 - \theta)(1/3)}{c_2^2} \text{ and } 15 = c_1^2 + c_2^2$$

## Competitive Equilibrium and Pareto Optimum

With  $\theta = 0.6$

$$\frac{\theta}{c_1^0} = \frac{1 - \theta}{c_2^0} \text{ and } 20 = c_1^0 + c_2^0$$

$$\frac{0.9\theta(2/3)}{c_1^1} = \frac{0.9(1 - \theta)(2/3)}{c_2^1} \text{ and } 25 = c_1^1 + c_2^1$$

$$\frac{0.9\theta(1/3)}{c_1^2} = \frac{0.9(1 - \theta)(1/3)}{c_2^2} \text{ and } 15 = c_1^2 + c_2^2$$

are satisfied with  $c_1^0 = 12$ ,  $c_2^0 = 8$ ,  $c_1^1 = 15$ ,  $c_1^2 = 10$ ,  $c_1^2 = 9$ , and  $c_2^2 = 6$ . To match the CE in which type 1 investors are wealthier, the social planner must give the utility of type 1 investors a higher weight  $\theta$ .

## Competitive Equilibrium and Pareto Optimum

The welfare theorems point to an easier way to find equilibria for A-D economies.

Start by solving a social planner's problem for a Pareto optimal allocation. There is only one optimization problem to solve and there are no prices involved, so this tends to be easier.

Then, given the Pareto allocation, “back out” the contingent claims prices using

$$q^i = \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \text{ for all } i = 1, 2, \dots, N \text{ and any } j = 1, 2, \dots, K$$

## Risk Sharing

We can use this approach to study how investors optimally share risk, keeping in mind that in the competitive equilibrium of an A-D economy they will use financial markets to do so,

Suppose once more that the economy consists of two types of investors – type 1 and 2 – in equal numbers.

Suppose again that there are only two possible states at  $t = 1$ : state 1, which occurs with probability  $\pi_1$ , and state 2, which occurs with probability  $\pi_2 = 1 - \pi_1$ .

## Risk Sharing

Aggregate endowments are  $w^0$  at  $t = 0$ ,  $w^1$  in state 1 at  $t = 1$ , and  $w^2$  in state 2 at  $t = 1$ .

The agent types both have expected utility, but may differ in terms of their Bernoulli utility functions and hence in terms of their risk aversion:

$$u_j(c_j^0) + \beta[\pi_1 u_j(c_j^1) + \pi_2 u_j(c_j^2)]$$

## Risk Sharing

The social planner chooses  $c_1^0$ ,  $c_2^0$ ,  $c_1^1$ ,  $c_2^1$ ,  $c_1^2$ , and  $c_2^2$  to maximize

$$\begin{aligned} & \theta \{ u_1(c_1^0) + \beta[\pi_1 u_1(c_1^1) + \pi_2 u_1(c_1^2)] \} \\ & + (1 - \theta) \{ u_2(c_2^0) + \beta[\pi_1 u_2(c_2^1) + \pi_2 u_2(c_2^2)] \} \end{aligned}$$

subject to the aggregate resource constraints

$$w^0 \geq c_1^0 + c_2^0$$

$$w^1 \geq c_1^1 + c_2^1$$

$$w^2 \geq c_1^2 + c_2^2$$

## Risk Sharing

The Lagrangian

$$\begin{aligned} L = & \theta \{ u_1(c_1^0) + \beta[\pi_1 u_1(c_1^1) + \pi_2 u_1(c_1^2)] \} \\ & + (1 - \theta) \{ u_2(c_2^0) + \beta[\pi_1 u_2(c_2^1) + \pi_2 u_2(c_2^2)] \} \\ & + \lambda^0 (w^0 - c_1^0 - c_2^0) + \lambda^1 (w^1 - c_1^1 - c_2^1) \\ & + \lambda^2 (w^2 - c_1^2 - c_2^2) \end{aligned}$$

leads to the first-order conditions

$$\theta u_1'(c_1^0) = \lambda^0 \text{ and } (1 - \theta) u_2'(c_2^0) = \lambda^0$$

$$\theta \beta \pi_1 u_1'(c_1^1) = \lambda^1 \text{ and } (1 - \theta) \beta \pi_1 u_2'(c_2^1) = \lambda^1$$

$$\theta \beta \pi_2 u_1'(c_1^2) = \lambda^2 \text{ and } (1 - \theta) \beta \pi_2 u_2'(c_2^2) = \lambda^2$$



## Risk Sharing

$$\theta u'_1(c_1^0) = \lambda^0 \text{ and } (1 - \theta)u'_2(c_2^0) = \lambda^0$$

$$\theta\beta\pi_1 u'_1(c_1^1) = \lambda^1 \text{ and } (1 - \theta)\beta\pi_1 u'_2(c_2^1) = \lambda^1$$

$$\theta\beta\pi_2 u'_1(c_1^2) = \lambda^2 \text{ and } (1 - \theta)\beta\pi_2 u'_2(c_2^2) = \lambda^2$$

Let's focus on the last two of these optimality conditions, since they show how the investors share risk across states 1 and 2 at  $t = 1$ :

$$\theta\beta\pi_1 u'_1(c_1^1) = (1 - \theta)\beta\pi_1 u'_2(c_2^1)$$

$$\theta\beta\pi_2 u'_1(c_1^2) = (1 - \theta)\beta\pi_2 u'_2(c_2^2)$$

## Risk Sharing

Using the aggregate resource constraints

$$\theta\beta\pi_1 u'_1(c_1^1) = (1 - \theta)\beta\pi_1 u'_2(c_2^1)$$

$$\theta\beta\pi_2 u'_1(c_1^2) = (1 - \theta)\beta\pi_2 u'_2(c_2^2)$$

imply

$$\theta u'_1(c_1^1) = (1 - \theta)u'_2(w^1 - c_1^1)$$

$$\theta u'_1(c_1^2) = (1 - \theta)u'_2(w^2 - c_1^2)$$

since  $c_2^1 = w^1 - c_1^1$  and  $c_2^2 = w^2 - c_1^2$

## Risk Sharing

$$\theta u'_1(c_1^1) = (1 - \theta)u'_2(w^1 - c_1^1)$$

$$\theta u'_1(c_1^2) = (1 - \theta)u'_2(w^2 - c_1^2)$$

Note, first, that when  $w^1 = w^2$ , so that there is no aggregate risk, these two equations are **exactly** the same.

In this case, we must have  $c_1^1 = c_1^2$  and  $c_2^1 = c_2^2$ . For each type of agent, consumption in state 1 is the same as consumption in state 2.

## Risk Sharing

With  $w^1 = w^2$ , we have  $c_1^1 = c_1^2$  and  $c_2^1 = c_2^2$ . For each type of agent, consumption in state 1 is the same as consumption in state 2.

The two welfare theorems then imply that in **any** competitive equilibrium without aggregate risk, agents will use financial markets to diversify away **all** idiosyncratic risk in their individual endowments.

This result holds true even when individual investors differ in their risk aversion.

## Risk Sharing

$$\theta u'_1(c_1^1) = (1 - \theta) u'_2(w^1 - c_1^1)$$

$$\theta u'_1(c_1^2) = (1 - \theta) u'_2(w^2 - c_1^2)$$

Next, suppose that  $w^1 \neq w^2$ , so that there is aggregate risk.

In this case, the two equations are different. At least one of the two types of investors will have consumption that differs across states 1 and 2 at  $t = 1$ . Aggregate risk, by definition, cannot be diversified away.

## Risk Sharing

$$\theta u'_1(c_1^1) = (1 - \theta) u'_2(w^1 - c_1^1)$$

$$\theta u'_1(c_1^2) = (1 - \theta) u'_2(w^2 - c_1^2)$$

An interesting special case arises when the two investor types have Bernoulli utility functions of the same CRRA form so that

$$u_j(c) = \frac{c^{1-\gamma} - 1}{1-\gamma} \text{ and } u'_j(c) = c^{-\gamma}$$

with  $\gamma > 0$  for  $j = 1$  and  $j = 2$ .

## Risk Sharing

When investors have identical CRRA utility functions

$$\theta u'_1(c_1^1) = (1 - \theta) u'_2(w^1 - c_1^1)$$

$$\theta u'_1(c_1^2) = (1 - \theta) u'_2(w^2 - c_1^2)$$

specialize to

$$\theta (c_1^1)^{-\gamma} = (1 - \theta) (w^1 - c_1^1)^{-\gamma}$$

$$\theta (c_1^2)^{-\gamma} = (1 - \theta) (w^2 - c_1^2)^{-\gamma}$$

## Risk Sharing

$$\theta(c_1^1)^{-\gamma} = (1 - \theta)(w^1 - c_1^1)^{-\gamma}$$

$$\theta(c_1^2)^{-\gamma} = (1 - \theta)(w^2 - c_1^2)^{-\gamma}$$

imply

$$\theta(w^1 - c_1^1)^\gamma = (1 - \theta)(c_1^1)^\gamma$$

$$\theta(w^2 - c_1^2)^\gamma = (1 - \theta)(c_1^2)^\gamma$$

or

$$\theta^{1/\gamma}(w^1 - c_1^1) = (1 - \theta)^{1/\gamma}c_1^1$$

$$\theta^{1/\gamma}(w^2 - c_1^2) = (1 - \theta)^{1/\gamma}c_1^2$$



## Risk Sharing

$$\theta^{1/\gamma}(w^1 - c_1^1) = (1 - \theta)^{1/\gamma}c_1^1$$

$$\theta^{1/\gamma}(w^2 - c_1^2) = (1 - \theta)^{1/\gamma}c_1^2$$

can be solved for

$$c_1^1 = \left[ \frac{\theta^{1/\gamma}}{\theta^{1/\gamma} + (1 - \theta)^{1/\gamma}} \right] w^1 \text{ and } c_1^2 = \left[ \frac{\theta^{1/\gamma}}{\theta^{1/\gamma} + (1 - \theta)^{1/\gamma}} \right] w^2$$

$$c_2^1 = \left\{ 1 - \left[ \frac{\theta^{1/\gamma}}{\theta^{1/\gamma} + (1 - \theta)^{1/\gamma}} \right] \right\} w^1 = \left[ \frac{(1 - \theta)^{1/\gamma}}{\theta^{1/\gamma} + (1 - \theta)^{1/\gamma}} \right] w^1$$

and

$$c_2^2 = \left\{ 1 - \left[ \frac{\theta^{1/\gamma}}{\theta^{1/\gamma} + (1 - \theta)^{1/\gamma}} \right] \right\} w^2 = \left[ \frac{(1 - \theta)^{1/\gamma}}{\theta^{1/\gamma} + (1 - \theta)^{1/\gamma}} \right] w^2$$

## Risk Sharing

$$c_1^1 = \left[ \frac{\theta^{1/\gamma}}{\theta^{1/\gamma} + (1-\theta)^{1/\gamma}} \right] w^1 \quad \text{and} \quad c_1^2 = \left[ \frac{\theta^{1/\gamma}}{\theta^{1/\gamma} + (1-\theta)^{1/\gamma}} \right] w^2$$

$$c_2^1 = \left[ \frac{(1-\theta)^{1/\gamma}}{\theta^{1/\gamma} + (1-\theta)^{1/\gamma}} \right] w^1 \quad \text{and} \quad c_2^2 = \left[ \frac{(1-\theta)^{1/\gamma}}{\theta^{1/\gamma} + (1-\theta)^{1/\gamma}} \right] w^2$$

In the special case where all individual investors have the same CRRA utility function, they share aggregate risk equally, in the sense that each consumes a fixed fraction of the aggregate endowment in each possible state at  $t = 1$ .

## Risk Sharing

And when  $\gamma = 1$ ,

$$c_1^1 = \left[ \frac{\theta^{1/\gamma}}{\theta^{1/\gamma} + (1-\theta)^{1/\gamma}} \right] w^1 \text{ and } c_1^2 = \left[ \frac{\theta^{1/\gamma}}{\theta^{1/\gamma} + (1-\theta)^{1/\gamma}} \right] w^2$$

$$c_2^1 = \left[ \frac{(1-\theta)^{1/\gamma}}{\theta^{1/\gamma} + (1-\theta)^{1/\gamma}} \right] w^1 \text{ and } c_2^2 = \left[ \frac{(1-\theta)^{1/\gamma}}{\theta^{1/\gamma} + (1-\theta)^{1/\gamma}} \right] w^2$$

simplify even further to

$$c_1^1 = \theta w^1 \text{ and } c_1^2 = \theta w^2$$

$$c_2^1 = (1-\theta)w^1 \text{ and } c_2^2 = (1-\theta)w^2$$

## Risk Sharing

	$t = 0$	$t = 1$ state 1	$t = 1$ state 2
Endowments			
type 1	$w_1^0 = 8$	$w_1^1 = 20$	$w_1^2 = 13$
type 2	$w_2^0 = 12$	$w_2^1 = 5$	$w_2^2 = 2$
Consumptions			
type 1	$c_1^0 = 12$	$c_1^1 = 15$	$c_1^2 = 9$
type 2	$c_2^0 = 8$	$c_2^1 = 10$	$c_2^2 = 6$

The allocations from our numerical example have this special property: the type 1 investor always consumes  $\theta = 0.6$  (60 percent) of the aggregate endowment.

## Risk Sharing

Hence, the A-D model echoes the CAPM in another way: by emphasizing the difference between idiosyncratic risk, which can be diversified away in financial markets, and aggregate risk, which cannot be.

Once again, however, the A-D model draws explicit connections between risk in financial markets and risk in the economy as a whole.

## Equilibrium and No-Arbitrage

The A-D model is an explicit **equilibrium** model of asset prices.

Through the equilibrium condition

$$q^i = \frac{\beta \pi_i u_j'(c_j^i)}{u_j'(c_j^0)}$$

which must hold for all states  $i = 1, 2, \dots, N$  and all investors  $j = 1, 2, \dots, K$ , the A-D model links asset prices to aggregate, undiversifiable risk in the economy as a whole.

## Equilibrium and No-Arbitrage

The A-D model's generality has been both a strength and a weakness.

It is a strength, because the A-D model makes no special assumptions about investors' preferences or the distribution of asset returns.

But it is also a weakness, since it is difficult to see, at least at first, how it can be applied to think about assets that are actually traded in financial markets, like stocks, bonds, and options.

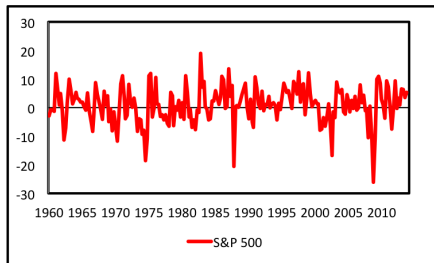
## Equilibrium and No-Arbitrage

Further progress with the A-D model can be made along two dimensions:

1. The Consumption Capital Asset Pricing Model (CCAPM) is a special case of the A-D model that adds further assumptions in order to get more specific results and to draw deeper links between the A-D model and the traditional CAPM.
2. The A-D model can also be used as a **no-arbitrage** theory of asset pricing. Through this approach, we can use existing assets to make inferences about what contingent claim prices should be, then use those contingent claim prices to price other assets as bundles of contingent claims, including assets like options with obviously non-normally distributed returns.

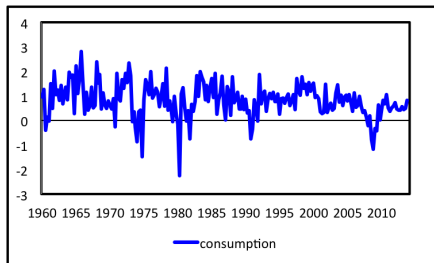


## Equilibrium and No-Arbitrage



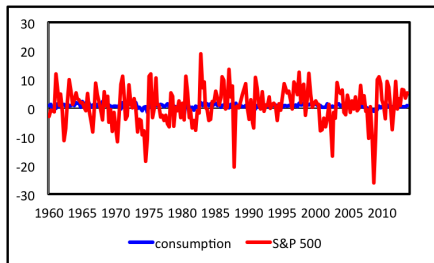
The inflation-adjusted growth rate of the Standard and Poor's index of stock prices is very volatile.

## Equilibrium and No-Arbitrage



So while real (inflation-adjusted) consumption growth does seem to be related to changes in stock prices . . .

# Equilibrium and No-Arbitrage



... stock prices seem much too volatile relative to consumption, compared to what the equilibrium version of Arrow-Debreu theory would predict.

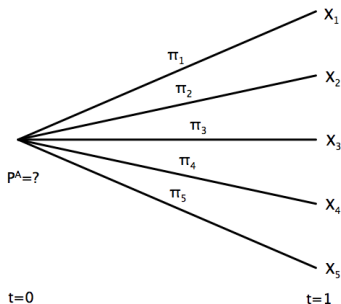
## Euler Equations

Before moving on, it will be useful to use a no-arbitrage argument to derive an equation that will lie at the heart of the CCAPM.

Consider an asset that, unlike a contingent claim, delivers payoffs in all  $N$  states of the world at  $t = 1$ .

Let  $\tilde{X}$  denote the random payoff as it appears to investors at  $t = 0$ , and let  $X_i$  denote more specifically the payoff made in each state  $i = 1, 2, \dots, N$  at  $t = 1$ .

# Euler Equations



The random payoff  $\tilde{X}$  equals  $X_i$  in each state  $i = 1, 2, 3, 4, 5$ .

## Euler Equations

The payoffs from this asset can be replicated by purchasing a bundle of contingent claims:

$X_1$  contingent claims for state 1

$X_2$  contingent claims for state 2

...

$X_N$  contingent claims for state N

## Euler Equations

The payoffs from this asset can be replicated by purchasing a bundle of contingent claims:

$X_1$  contingent claims for state 1 at cost  $q^1 X_1$

$X_2$  contingent claims for state 2 at cost  $q^2 X_2$

...

$X_N$  contingent claims for state N at cost  $q^N X_N$

## Euler Equations

A no-arbitrage argument implies that the price of the asset must equal the price of all of the contingent claims in the equivalent bundle.

If the price of the asset was less than the price of the bundle of contingent claims, investors could profit by buying the asset and selling the bundle of claims.

If the price of the bundle of contingent claims was less than the price of the asset, investors could profit by buying the bundle of claims and selling the asset.



## Euler Equations

Hence, the asset price must be

$$P^A = q^1 X_1 + q^2 X_2 + \dots + q^N X_N = \sum_{i=1}^N q^i X_i$$

In an A-D equilibrium, however,

$$q^i = \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \text{ for all } i = 1, 2, \dots, N$$

must hold for all  $j = 1, 2, \dots, K$

## Euler Equations

Substitute the A-D equilibrium conditions

$$q^i = \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \text{ for all } i = 1, 2, \dots, N$$

into the no-arbitrage pricing condition

$$\begin{aligned} P^A &= \sum_{i=1}^N q^i X_i \\ &= \sum_{i=1}^N \left[ \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \right] X_i \end{aligned}$$

## Euler Equations

$$P_A = \sum_{i=1}^N \left[ \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \right] X_i$$

implies

$$u'_j(c_j^0) P_A = \beta \sum_{i=1}^N \pi_i u'_j(c_j^i) X_i$$

or, using the definition of an expected value,

$$u'_j(c_j^0) P_A = \beta E[u'_j(\tilde{c}_j) \tilde{X}]$$

where  $\tilde{c}_j$  is the random value of period  $t = 1$  consumption for investor  $j$ .

## Euler Equations

Equivalently, if we define the **return** on the asset between  $t = 0$  and state  $i$  at  $t = 1$  as  $R_i = X_i/P^A$ , then

$$P_A = \sum_{i=1}^N \left[ \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \right] X_i$$

implies

$$1 = \beta \sum_{i=1}^N \left[ \frac{\pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \right] R_i$$

$$1 = \beta E \left[ \left( \frac{u'_j(\tilde{c}_j)}{u'_j(c_j^0)} \right) \tilde{R} \right]$$

where  $\tilde{R}$  is the random asset return between  $t = 0$  and  $t = 1$ .

## Euler Equations

Yet another representation of the same basic condition can be obtained by noting that

$$m_j^i = \frac{\beta u_j'(c_j^i)}{u_j'(c_j^0)}$$

measures investor  $j$ 's marginal rate of substitution between  $t = 0$  and state  $i$  at  $t = 1$ . With this in mind, define investor  $j$ 's random **intertemporal marginal rate of substitution** as

$$\tilde{m}_j = \frac{\beta u_j'(\tilde{c}_j)}{u_j'(c_j^0)}$$

# Euler Equations

The definition

$$\tilde{m}_j = \frac{\beta u'_j(\tilde{c}_j)}{u'_j(c_j^0)}$$

allows

$$1 = \beta E \left[ \left( \frac{u'_j(\tilde{c}_j)}{u'_j(c_j^0)} \right) \tilde{R} \right]$$

to be written more compactly as

$$1 = E(\tilde{m}_j \tilde{R})$$

## Euler Equations

Notice that in the case without uncertainty

$$1 = E(\tilde{m}_j \tilde{R})$$

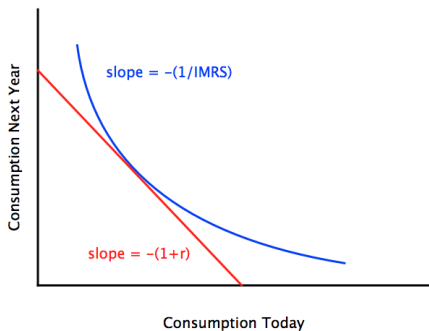
becomes

$$\frac{1}{R} = m_j$$

$$\frac{1}{1+r} = \frac{\beta u'_j(c_j^1)}{u'_j(c_j^0)}$$

where  $r$  is the interest rate between  $t = 0$  and  $t = 1$  and  $c_j^1$  is investor  $j$ 's consumption for sure at  $t = 1$ .

# Euler Equations



$1 = E(\tilde{m}_j \tilde{R})$  generalizes Irving Fisher's theory of interest to the case with uncertainty.



## Euler Equations

These three equivalent relationships

$$P_A = \sum_{i=1}^N \left[ \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \right] X_i$$

$$1 = \beta E \left[ \left( \frac{u'_j(\tilde{c}_j)}{u'_j(c_j^0)} \right) \tilde{R} \right]$$

$$1 = E(\tilde{m}_j \tilde{R})$$

are often referred to as **Euler equations**, named after Leonhard Euler (Switzerland, 1707-1783), who derived similar equations in developing a mathematical theory of optimization that he called the **calculus of variations**.

# Euler Equations

Versions of the Euler equations

$$P_A = \sum_{i=1}^N \left[ \frac{\beta \pi_i u'_j(c_j^i)}{u'_j(c_j^0)} \right] X_i$$

$$1 = \beta E \left[ \left( \frac{u'_j(\tilde{c}_j)}{u'_j(c_j^0)} \right) \tilde{R} \right]$$

$$1 = E(\tilde{m}_j \tilde{R})$$

will appear again and play a key role, both in our analysis of the CCAPM and in no-arbitrage pricing theories.