

# ECON 337901

# FINANCIAL ECONOMICS

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## 5 Risk Aversion and Investment Decisions

- A Risk Aversion and Portfolio Allocation
- B Portfolios, Risk Aversion, and Wealth
- C Risk Aversion and Saving Behavior
- D Separating Risk and Time Preferences

## Risk Aversion and Portfolio Allocation

Let's now put our framework of decision-making under uncertainty to use.

Consider a risk-averse investor with vN-M expected utility who divides his or her initial wealth  $Y_0$  into an amount  $a$  allocated to a risky asset – say, the stock market – and an amount  $Y_0 - a$  allocated to a safe asset – say, a bank account or a government bond.

## Risk Aversion and Portfolio Allocation

$Y_0$  = initial wealth

$a$  = amount allocated to stocks

$\tilde{r}$  = random return on stocks

$r_f$  = risk-free return

$\tilde{Y}_1$  = terminal wealth

$$\begin{aligned}\tilde{Y}_1 &= (1 + r_f)(Y_0 - a) + a(1 + \tilde{r}) \\ &= Y_0(1 + r_f) + a(\tilde{r} - r_f)\end{aligned}$$

## Risk Aversion and Portfolio Allocation

The investor chooses  $a$  to maximize expected utility:

$$\max_a E[u(\tilde{Y}_1)] = \max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

If the investor is risk-averse,  $u$  is concave and the first-order condition for this unconstrained optimization problem, found by differentiating the objective function by the choice variable and equating to zero, is both a necessary and sufficient condition for the value  $a^*$  of  $a$  that solves the problem.

## Risk Aversion and Portfolio Allocation

The investor's problem is

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

The first-order condition is

$$E\{u'[Y_0(1 + r_f) + a^*(\tilde{r} - r_f)](\tilde{r} - r_f)\} = 0.$$

Note: we are allowing the investor to sell stocks short ( $a^* < 0$ ) or to buy stocks on margin ( $a^* > Y_0$ ) if he or she desires.

## Risk Aversion and Portfolio Allocation

**Theorem** If the Bernoulli utility function  $u$  is increasing and concave, then

$$a^* > 0 \text{ if and only if } E(\tilde{r}) > r_f$$

$$a^* = 0 \text{ if and only if } E(\tilde{r}) = r_f$$

$$a^* < 0 \text{ if and only if } E(\tilde{r}) < r_f$$

Thus, a risk-averse investor will **always** allocate at least some funds to the stock market if the expected return on stocks exceeds the risk-free rate.

## Risk Aversion and Portfolio Allocation

To prove the theorem, let

$$W(a) = E\{u'[Y_0(1 + r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)\},$$

so that the investor's first-order condition can be written more compactly as

$$W(a^*) = 0.$$



## Risk Aversion and Portfolio Allocation

Next, note that with

$$W(a) = E\{u'[Y_0(1 + r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)\},$$

it follows that

$$W'(a) = E\{u''[Y_0(1 + r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)^2\} < 0$$

since  $u$  is concave. This means that  $W$  is a decreasing function of  $a$ .

## Risk Aversion and Portfolio Allocation

Finally, note that with

$$W(a) = E\{u'[Y_0(1 + r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)\},$$

$$\begin{aligned}W(0) &= E\{u'[Y_0(1 + r_f) + 0(\tilde{r} - r_f)](\tilde{r} - r_f)\} \\&= E\{u'[Y_0(1 + r_f)](\tilde{r} - r_f)\} \\&= u'[Y_0(1 + r_f)]E(\tilde{r} - r_f) \\&= u'[Y_0(1 + r_f)][E(\tilde{r}) - r_f].\end{aligned}$$

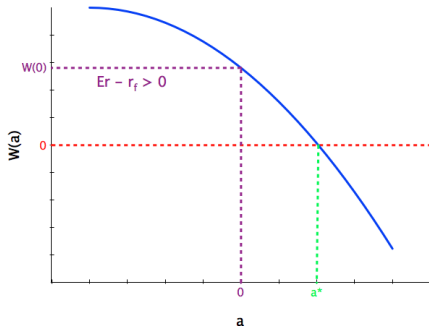
Since  $u$  is increasing, this means that  $W(0)$  has the same sign as  $E(\tilde{r}) - r_f$ .

# Risk Aversion and Portfolio Allocation

We now know that:

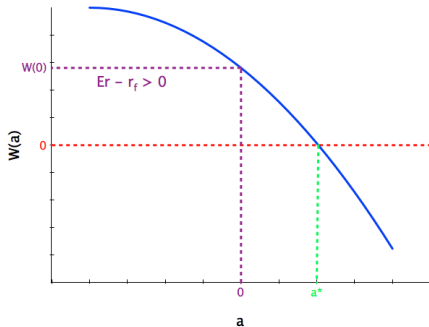
1.  $W(a)$  is a decreasing function
2.  $W(0)$  has the same sign as  $E(\tilde{r}) - r_f$ .
3.  $W(a^*) = 0$

# Risk Aversion and Portfolio Allocation



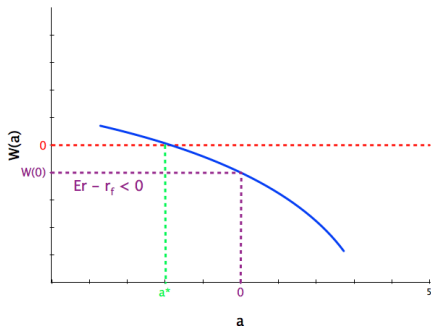
$E(\tilde{r}) - r_f > 0$  implies that  $W(0) > 0$ , and since  $W$  is decreasing,  $W(a^*) = 0$  implies that  $a^* > 0$ .

## Risk Aversion and Portfolio Allocation



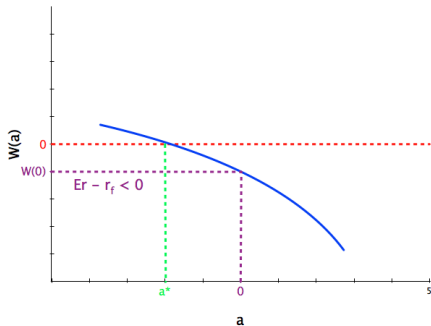
Since  $W$  is decreasing,  $W(a^*) = 0$  and  $a^* > 0$  imply that  $W(0) > 0$ . And since  $W(0)$  has the same sign as  $E(\tilde{r}) - r_f$ ,  $E(\tilde{r}) - r_f > 0$ .

# Risk Aversion and Portfolio Allocation



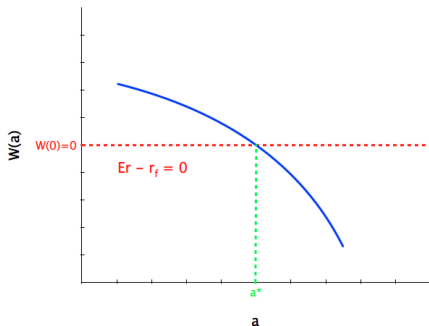
$E(\tilde{r}) - r_f < 0$  implies that  $W(0) < 0$ , and since  $W$  is decreasing,  $W(a^*) = 0$  implies that  $a^* < 0$ .

## Risk Aversion and Portfolio Allocation



Since  $W$  is decreasing,  $W(a^*) = 0$  and  $a^* < 0$  imply that  $W(0) < 0$ . And since  $W(0)$  has the same sign as  $E(\tilde{r}) - r_f$ ,  $E(\tilde{r}) - r_f < 0$ .

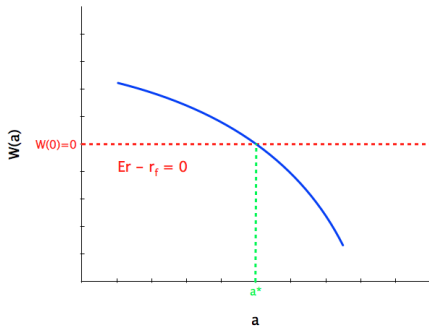
# Risk Aversion and Portfolio Allocation



$E(\tilde{r}) - r_f = 0$  implies that  $W(0) = 0$ , and since  $W$  is decreasing,  $W(a^*) = 0$  implies that  $a^* = 0$ .



## Risk Aversion and Portfolio Allocation



Since  $W$  is decreasing,  $W(a^*) = 0$  and  $a^* = 0$  imply that  $W(0) = 0$ . And since  $W(0)$  has the same sign as  $E(\tilde{r}) - r_f$ ,  $E(\tilde{r}) - r_f = 0$ .

## Risk Aversion and Portfolio Allocation

**Theorem** If the Bernoulli utility function  $u$  is increasing and concave, then

$$a^* > 0 \text{ if and only if } E(\tilde{r}) > r_f$$

$$a^* = 0 \text{ if and only if } E(\tilde{r}) = r_f$$

$$a^* < 0 \text{ if and only if } E(\tilde{r}) < r_f$$

Thus, a risk-averse investor will **always** allocate at least some funds to the stock market if the expected return on stocks exceeds the risk-free rate.

## Risk Aversion and Portfolio Allocation

Danthine and Donaldson (3rd ed., p.41) report that in the United States, 1889-2010, average real (inflation-adjusted) returns on stocks and risk-free bonds are

$$E(\tilde{r}) = 0.075 \text{ (7.5 percent per year)}$$

$$r_f = 0.011 \text{ (1.1 percent per year)}$$

The **equity risk premium** of  $E(\tilde{r}) - r_f = 0.064$  (6.4 percent) is not only positive, it is huge. The implication of the theory is that all investors, even the most risk averse, should have some money invested in the stock market.

## Risk Aversion and Portfolio Allocation

As an example, suppose  $u(Y) = \ln(Y)$ , as suggested by Daniel Bernoulli. Recall that for this utility function,  $u'(Y) = 1/Y$ . Then assume that stock returns can either be good or bad:

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1 - \pi \end{cases}$$

where  $r_G > r_f > r_B$  defines the “good” and “bad” states and

$$\pi r_G + (1 - \pi)r_B > r_f,$$

so that  $E(\tilde{r}) > r_f$  and the investor will choose  $a^* > 0$ .

# Risk Aversion and Portfolio Allocation

The problem

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

specializes to

$$\begin{aligned} \max_a \quad & \pi \ln[Y_0(1 + r_f) + a(r_G - r_f)] \\ & + (1 - \pi) \ln[Y_0(1 + r_f) + a(r_B - r_f)] \end{aligned}$$

## Risk Aversion and Portfolio Allocation

The problem

$$\begin{aligned} \max_a \quad & \pi \ln[Y_0(1 + r_f) + a(r_G - r_f)] \\ & + (1 - \pi) \ln[Y_0(1 + r_f) + a(r_B - r_f)] \end{aligned}$$

has first-order condition

$$\frac{\pi(r_G - r_f)}{Y_0(1 + r_f) + a^*(r_G - r_f)} + \frac{(1 - \pi)(r_B - r_f)}{Y_0(1 + r_f) + a^*(r_B - r_f)} = 0.$$

## Risk Aversion and Portfolio Allocation

$$\frac{\pi(r_G - r_f)}{Y_0(1 + r_f) + a^*(r_G - r_f)} + \frac{(1 - \pi)(r_B - r_f)}{Y_0(1 + r_f) + a^*(r_B - r_f)} = 0$$

$$\begin{aligned} & \pi(r_G - r_f)[Y_0(1 + r_f) + a^*(r_B - r_f)] \\ = & -(1 - \pi)(r_B - r_f)[Y_0(1 + r_f) + a^*(r_G - r_f)] \end{aligned}$$

$$\begin{aligned} & a^*(r_G - r_f)(r_B - r_f) \\ = & -Y_0(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)] \end{aligned}$$

## Risk Aversion and Portfolio Allocation

$$\begin{aligned} & a^*(r_G - r_f)(r_B - r_f) \\ = & -Y_0(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)] \end{aligned}$$

implies

$$\frac{a^*}{Y_0} = -\frac{(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

which is positive, since  $r_G > r_f > r_B$  and

$$E(\tilde{r}) - r_f = \pi(r_G - r_f) + (1 - \pi)(r_B - r_f) > 0.$$



## Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = - \frac{(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

In this case,  $a^*$ :

Rises proportionally with  $Y_0$ .

Increases as  $E(\tilde{r}) - r_f$  rises.

Falls as  $r_G$  and  $r_B$  move farther away from  $r_f$ , holding  $E(\tilde{r})$  constant; that is, in response to a mean preserving spread.

## Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = - \frac{(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

$r_f$	$r_G$	$r_B$	$\pi$	$E(\tilde{r})$	$a^*/Y_0$
0.05	0.40	-0.20	0.50	0.10	0.60
0.05	0.30	-0.10	0.50	0.10	1.40
0.05	0.40	-0.20	0.75	0.25	2.40

The fraction of initial wealth allocated to stocks rises when stocks become less risky or pay higher expected returns.

## Risk Aversion and Portfolio Allocation

Before moving on, return to the general problem

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

but assume now that the investor is **risk-neutral**, with

$$u(Y) = \alpha Y + \beta,$$

and  $\alpha > 0$ , so that more wealth is preferred to less.

## Risk Aversion and Portfolio Allocation

The risk-neutral investor solves

$$\begin{aligned} & \max_a E\{u[Y_0(1+r_f) + a(\tilde{r} - r_f)]\} \\ = & \max_a E\{\alpha[Y_0(1+r_f) + a(\tilde{r} - r_f)] + \beta\} \\ = & \max_a \alpha\{Y_0(1+r_f) + a[E(\tilde{r}) - r_f]\} + \beta \end{aligned}$$

So long as  $E(\tilde{r}) - r_f > 0$ , the risk-neutral investor will choose  $a^*$  to be as large as possible, borrowing as much as he or she is allowed to in order to buy more stocks on margin.

## Portfolios, Risk Aversion, and Wealth

The previous examples call out for a more detailed analysis of how optimal portfolio allocation decisions, summarized by the value of  $a^*$  that solves

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

are influenced by the investor's degree of risk aversion and his or her level of wealth.

## Portfolios, Risk Aversion, and Wealth

The following result was proven by Kenneth Arrow in “The Theory of Risk Aversion,” published in the 1971 volume *Essays in the Theory of Risk-Bearing* and reprinted in 1983 in volume 3 of the *Collected Papers of Kenneth J. Arrow* (Harvard University Press).

**Theorem** Consider two investors,  $i = 1$  and  $i = 2$ , and suppose that for all wealth levels  $Y$ ,  $R_A^1(Y) > R_A^2(Y)$ , where  $R_A^i(Y)$  is investor  $i$ 's coefficient of absolute risk aversion. Then  $a_1^*(Y) < a_2^*(Y)$ , where  $a_i^*(Y)$  is amount allocated by investor  $i$  to stocks when he or she has initial wealth  $Y$ .

## Portfolios, Risk Aversion, and Wealth

Recall that the coefficients of absolute and relative risk aversion are

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} \text{ and } R_R(Y) = -\frac{Y u''(Y)}{u'(Y)}.$$

Thus

$$R_A^1(Y) > R_A^2(Y) \text{ or } -\frac{u_1''(Y)}{u_1'(Y)} > -\frac{u_2''(Y)}{u_2'(Y)}$$

implies

$$-\frac{Y u_1''(Y)}{u_1'(Y)} > -\frac{Y u_2''(Y)}{u_2'(Y)} \text{ or } R_R^1(Y) > R_R^2(Y).$$

## Portfolios, Risk Aversion, and Wealth

Arrow's result applies equally well to relative risk aversion:

**Theorem** Consider two investors,  $i = 1$  and  $i = 2$ , and suppose that for all wealth levels  $Y$ ,  $R_R^1(Y) > R_R^2(Y)$ , where  $R_R^i(Y)$  is investor  $i$ 's coefficient of relative risk aversion. Then  $a_1^*(Y) < a_2^*(Y)$ , where  $a_i^*(Y)$  is amount allocated by investor  $i$  to stocks when he or she has initial wealth  $Y$ .



## Portfolios, Risk Aversion, and Wealth

Let's test Arrow's proposition out, by generalizing our previous example with logarithmic utility to the case where

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma},$$

with  $\gamma > 0$ . For this Bernoulli utility function, the coefficient of relative risk aversion is constant and equal to  $\gamma$ . The specific setting  $\gamma = 1$  takes us back to the case with logarithmic utility.

## Portfolios, Risk Aversion, and Wealth

Hence, in this extended example,

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma} \text{ implies } u'(Y) = Y^{-\gamma} = \frac{1}{Y^\gamma}.$$

and stock returns can either be good or bad

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1 - \pi \end{cases}$$

where  $r_G > r_f > r_B$  defines the “good” and “bad” states and

$$\pi r_G + (1 - \pi)r_B > r_f,$$

so that  $E(\tilde{r}) > r_f$  and the investor will choose  $a^* > 0$ .

## Portfolios, Risk Aversion, and Wealth

With CRRA (constant relative risk aversion) utility and two states for  $\tilde{r}$ , the problem

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

specializes to

$$\begin{aligned} \max_a \quad & \pi \left\{ \frac{[Y_0(1 + r_f) + a(r_G - r_f)]^{1-\gamma} - 1}{1 - \gamma} \right\} \\ & + (1 - \pi) \left\{ \frac{[Y_0(1 + r_f) + a(r_B - r_f)]^{1-\gamma} - 1}{1 - \gamma} \right\} \end{aligned}$$

## Portfolios, Risk Aversion, and Wealth

The problem

$$\max_a \pi \left\{ \frac{[Y_0(1+r_f) + a(r_G - r_f)]^{1-\gamma} - 1}{1-\gamma} \right\} \\ + (1-\pi) \left\{ \frac{[Y_0(1+r_f) + a(r_B - r_f)]^{1-\gamma} - 1}{1-\gamma} \right\}$$

has first-order condition

$$\frac{\pi(r_G - r_f)}{[Y_0(1+r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1-\pi)(r_B - r_f)}{[Y_0(1+r_f) + a^*(r_B - r_f)]^\gamma} = 0.$$

## Portfolios, Risk Aversion, and Wealth

$$\frac{\pi(r_G - r_f)}{[Y_0(1 + r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1 - \pi)(r_B - r_f)}{[Y_0(1 + r_f) + a^*(r_B - r_f)]^\gamma} = 0$$

$$\begin{aligned} & \pi(r_G - r_f)[Y_0(1 + r_f) + a^*(r_B - r_f)]^\gamma \\ = & (1 - \pi)(r_f - r_B)[Y_0(1 + r_f) + a^*(r_G - r_f)]^\gamma \end{aligned}$$

$$\begin{aligned} & [\pi(r_G - r_f)]^{1/\gamma} [Y_0(1 + r_f) + a^*(r_B - r_f)] \\ = & [(1 - \pi)(r_f - r_B)]^{1/\gamma} [Y_0(1 + r_f) + a^*(r_G - r_f)] \end{aligned}$$

## Portfolios, Risk Aversion, and Wealth

$$\begin{aligned} & [\pi(r_G - r_f)]^{1/\gamma} [Y_0(1 + r_f) + a^*(r_B - r_f)] \\ = & [(1 - \pi)(r_f - r_B)]^{1/\gamma} [Y_0(1 + r_f) + a^*(r_G - r_f)] \end{aligned}$$

$$\begin{aligned} & Y_0(1 + r_f) [\pi(r_G - r_f)]^{1/\gamma} + a^*(r_B - r_f) [\pi(r_G - r_f)]^{1/\gamma} \\ = & Y_0(1 + r_f) [(1 - \pi)(r_f - r_B)]^{1/\gamma} \\ & + a^*(r_G - r_f) [(1 - \pi)(r_f - r_B)]^{1/\gamma} \end{aligned}$$

$$\begin{aligned} & Y_0(1 + r_f) \{ [\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma} \} \\ = & a^* \{ (r_G - r_f) [(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B) [\pi(r_G - r_f)]^{1/\gamma} \} \end{aligned}$$

## Portfolios, Risk Aversion, and Wealth

$$\begin{aligned} & Y_0(1 + r_f)\{\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma}\} \\ = & a^*\{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}\} \end{aligned}$$

implies

$$\frac{a^*}{Y_0} = \frac{(1 + r_f)\{\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma}\}}{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

## Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1 + r_f) \{ [\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma} \}}{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

$\gamma$	$r_f$	$r_G$	$r_B$	$\pi$	$E(\tilde{r})$	$a^*/Y_0$
0.5	0.05	0.40	-0.20	0.50	0.10	1.20
1	0.05	0.40	-0.20	0.50	0.10	0.60
2	0.05	0.40	-0.20	0.50	0.10	0.30
3	0.05	0.40	-0.20	0.50	0.10	0.20
5	0.05	0.40	-0.20	0.50	0.10	0.12
10	0.05	0.40	-0.20	0.50	0.10	0.06



## Portfolios, Risk Aversion, and Wealth

$\gamma$	$r_f$	$r_G$	$r_B$	$\pi$	$E(\tilde{r})$	$a^*/Y_0$
0.5	0.05	0.40	-0.20	0.50	0.10	1.20
1	0.05	0.40	-0.20	0.50	0.10	0.60
2	0.05	0.40	-0.20	0.50	0.10	0.30
3	0.05	0.40	-0.20	0.50	0.10	0.20
5	0.05	0.40	-0.20	0.50	0.10	0.12
10	0.05	0.40	-0.20	0.50	0.10	0.06

Consistent with Arrow's theorem, higher coefficients of relative risk aversion are associated with smaller values of  $a^*$ .

## Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1 + r_f) \{ [\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma} \}}{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

Note that with constant relative risk aversion,  $a^*$  rises proportionally with wealth.

Two additional theorems, also proven by Arrow, tell us more about the relationship between  $a^*$  and wealth.

# Portfolios, Risk Aversion, and Wealth

**Theorem** Let  $a^*(Y_0)$  be the solution to

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}.$$

If  $u(Y)$  is such that

(a)  $R'_A(Y) < 0$  then  $\frac{da^*(Y_0)}{dY_0} > 0$

(b)  $R'_A(Y) = 0$  then  $\frac{da^*(Y_0)}{dY_0} = 0$

(c)  $R'_A(Y) > 0$  then  $\frac{da^*(Y_0)}{dY_0} < 0$

# Portfolios, Risk Aversion, and Wealth

Part (a)

$$R'_A(Y) < 0 \text{ then } \frac{da^*(Y_0)}{dY_0} > 0$$

describes the “normal” case where absolute risk aversion falls as wealth rises.

In this case, wealthier individuals allocate more wealth to stocks.

## Portfolios, Risk Aversion, and Wealth

Part (b)

$$R'_A(Y) = 0 \text{ then } \frac{da^*(Y_0)}{dY_0} = 0$$

means that investors with constant absolute risk aversion

$$u(Y) = -\frac{1}{\nu} e^{-\nu Y}$$

allocate a constant amount of wealth to stocks.

This may seem surprising, but it reflects that fact that absolute risk aversion describes preferences over bets of a given size ...

## Portfolios, Risk Aversion, and Wealth

Part (b)

$$R'_A(Y) = 0 \text{ then } \frac{da^*(Y_0)}{dY_0} = 0$$

means that investors with constant absolute risk aversion

$$u(Y) = -\frac{1}{\nu} e^{-\nu Y}$$

allocate a constant amount of wealth to stocks.

...so a CARA investor finds a bet of the ideal size and sticks with it, even when income increases.

# Portfolios, Risk Aversion, and Wealth

Part (c)

$$R'_A(Y) > 0 \text{ then } \frac{da^*(Y_0)}{dY_0} < 0$$

describes the case where absolute risk aversion rises as wealth rises.

The implication that wealthier individuals allocate less wealth to stocks makes this case seem less plausible.

## Portfolios, Risk Aversion, and Wealth

**Theorem** Let  $a^*(Y_0)$  be the solution to

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}.$$

If  $u(Y)$  is such that

- (a)  $R'_A(Y) < 0$  then  $\frac{da^*(Y_0)}{dY_0} > 0$
- (b)  $R'_A(Y) = 0$  then  $\frac{da^*(Y_0)}{dY_0} = 0$
- (c)  $R'_A(Y) > 0$  then  $\frac{da^*(Y_0)}{dY_0} < 0$

This result relates changes in **absolute** risk aversion to the **absolute** amount of wealth allocated to stocks.



## Portfolios, Risk Aversion, and Wealth

Consistent with our results with CRRA utility, the next result relates changes in **relative** risk aversion to changes in the **proportion** of wealth allocated to stocks.

Define the **elasticity** of the function  $a^*(Y_0)$  as

$$\eta = \frac{d \ln a^*(Y_0)}{d \ln Y_0} = \frac{Y_0}{a^*(Y_0)} \frac{da^*(Y_0)}{dY_0}$$

The elasticity measures the **percentage** change in  $a^*$  brought about by a percentage-point change in  $Y_0$ .

## Portfolios, Risk Aversion, and Wealth

**Theorem** Let  $a^*(Y_0)$  be the solution to

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}.$$

If  $u(Y)$  is such that

- (a)  $R'_R(Y) < 0$  then  $\eta > 1$
- (b)  $R'_R(Y) = 0$  then  $\eta = 1$
- (c)  $R'_R(Y) > 0$  then  $\eta < 1$

The theorem confirms what we know about CRRA utility: it implies that  $a^*$  rises proportionally with  $Y_0$ .

## Portfolios, Risk Aversion, and Wealth

With CRRA utility:

$$\frac{a^*}{Y_0} = K$$

where

$$K = \frac{(1 + r_f) \{ [\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma} \}}{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}.$$

Hence

$$\ln(a^*(Y_0)) = \ln(K) + \ln(Y_0)$$

and

$$\eta = \frac{d \ln a^*(Y_0)}{d \ln Y_0} = 1.$$

## Portfolios, Risk Aversion, and Wealth

**Theorem** Let  $a^*(Y_0)$  be the solution to

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}.$$

If  $u(Y)$  is such that

- (a)  $R'_R(Y) < 0$  then  $\eta > 1$
- (b)  $R'_R(Y) = 0$  then  $\eta = 1$
- (c)  $R'_R(Y) > 0$  then  $\eta < 1$

But the theorem extends the results to the cases of decreasing and increasing relative risk aversion.

## Risk Aversion and Saving Behavior

So far, we've assumed that investors only receive utility from the terminal value of their wealth, and asked how they should split their initial wealth – accumulated, presumably, through past saving – across risky and riskless assets in order to maximize the expected utility from terminal wealth.

Now, let's take the possibly random return on the investor's portfolio of assets as given, and ask how he or she should optimally determine savings under conditions of uncertainty.

## Risk Aversion and Saving Behavior

Suppose there are two periods,  $t = 0$  and  $t = 1$ , and let

$Y_0 =$  initial wealth

$s =$  amount saved in period  $t = 0$

$c_0 = Y_0 - s =$  amount consumed in period  $t = 0$

$\tilde{R} = 1 + \tilde{r} =$  random, gross return on savings

$\tilde{c}_1 = s\tilde{R} =$  amount consumed in period  $t = 1$

Suppose also that the investor has vN-M expected utility over consumption during periods  $t = 0$  and  $t = 1$  given by

$$u(c_0) + \beta E[u(\tilde{c}_1)] = u(Y_0 - s) + \beta E[u(s\tilde{R})],$$

where the **discount factor**  $\beta$  is a measure of patience.

## Risk Aversion and Saving Behavior

The solution to the investor's saving problem

$$\max_s u(Y_0 - s) + \beta E[u(s\tilde{R})]$$

is described by the first-order condition

$$-u'(Y_0 - s^*) + \beta E[u'(s^*\tilde{R})\tilde{R}] = 0$$

or

$$u'(Y_0 - s^*) = \beta E[u'(s^*\tilde{R})\tilde{R}]$$

## Risk Aversion and Saving Behavior

$$u'(Y_0 - s^*) = \beta E[u'(s^* \tilde{R}) \tilde{R}]$$

We can use this optimality condition to investigate how optimal saving  $s^*$  responds to an increase in risk, in the form of a mean preserving spread in the distribution of  $\tilde{R}$ . Intuitively, one might expect there to be two offsetting effects:

1. The riskier return will make saving less attractive and thereby reduce  $s^*$ .
2. The riskier return might lead to “precautionary saving” in order to cushion period  $t = 1$  consumption against the possibility of a bad output and thereby increase  $s^*$ .



## Risk Aversion and Saving Behavior

$$u'(Y_0 - s^*) = \beta E[u'(s^* \tilde{R}) \tilde{R}]$$

To see which of these two effects dominates, define

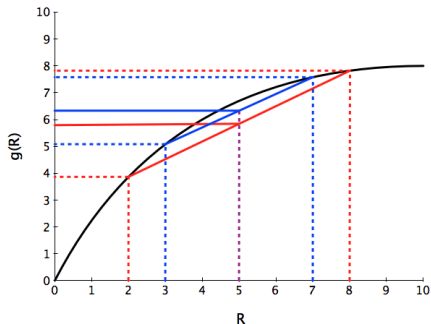
$$g(\tilde{R}) = u'(s^* \tilde{R}) \tilde{R}$$

so that the right-hand side becomes

$$\beta E[g(\tilde{R})].$$

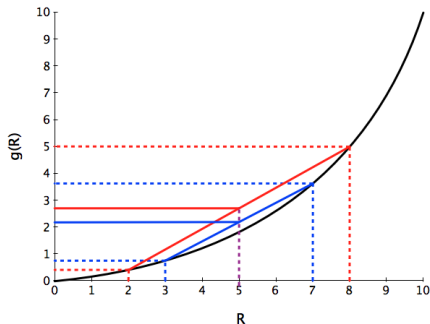
Jensen's inequality will imply that after a mean preserving spread the distribution of  $\tilde{R}$  in this expectation will fall if  $g$  is concave and rise if  $g$  is convex.

## Risk Aversion and Saving Behavior



When  $g$  is concave, a mean preserving spread in the distribution of  $\tilde{R}$  will decrease  $E[g(\tilde{R})]$ .

## Risk Aversion and Saving Behavior



When  $g$  is convex, a mean preserving spread in the distribution of  $\tilde{R}$  will increase  $E[g(\tilde{R})]$ .

# Risk Aversion and Saving Behavior

The definition

$$g(\tilde{R}) = u'(s^* \tilde{R}) \tilde{R}$$

suggests that the concavity or convexity of  $g$  will depend on the sign of the **third** derivative of  $u$ .

The product and chain rules for differentiation imply

$$g'(\tilde{R}) = u''(s^* \tilde{R}) s^* \tilde{R} + u'(s^* \tilde{R})$$

$$g''(\tilde{R}) = u'''(s^* \tilde{R}) (s^*)^2 \tilde{R} + u''(s^* \tilde{R}) s + u''(s^* \tilde{R}) s$$

## Risk Aversion and Saving Behavior

$$\begin{aligned}g''(\tilde{R}) &= u'''(s^* \tilde{R})(s^*)^2 \tilde{R} + u''(s^* \tilde{R})s + u''(s^* \tilde{R})s \\ &= u'''(s^* \tilde{R})(s^*)^2 \tilde{R} + 2u''(s^* \tilde{R})s\end{aligned}$$

implies that  $g''(\tilde{R})$  has the same sign as

$$u'''(s^* \tilde{R})s \tilde{R} + 2u''(s^* \tilde{R})$$

## Risk Aversion and Saving Behavior

To understand precautionary saving behavior, the concept of **prudence** is defined by Miles Kimball, "Precautionary Saving in the Small and in the Large," *Econometrica* Vol.58 (January 1990): pp.53-73.

Whereas risk aversion is summarized by the second derivative of the Bernoulli utility function  $u$ , prudence is summarized by the third derivative of  $u$ .

## Risk Aversion and Saving Behavior

Kimball defines the **coefficient of absolute prudence** as

$$P_A(Y) = -\frac{u'''(Y)}{u''(Y)}$$

and the **coefficient of relative prudence** as

$$P_R(Y) = -\frac{Y u'''(Y)}{u''(Y)}$$

thereby extending the analogous measures of absolute and relative risk aversion.

## Risk Aversion and Saving Behavior

Since  $g''(\tilde{R})$  has the same sign as

$$u'''(s^*\tilde{R})s\tilde{R} + 2u''(s^*\tilde{R})$$

or

$$u'''(Y)Y + 2u''(Y) = u''(Y) \left[ \frac{u'''(Y)Y}{u''(Y)} + 2 \right] = u''(Y)[2 - P_R(Y)]$$

$g''(\tilde{R})$  is positive if  $2 < P_R(Y)$

$g''(\tilde{R})$  is negative if  $2 > P_R(Y)$



## Risk Aversion and Saving Behavior

Hence, if  $2 < P_R(Y)$ , then  $g''(\tilde{R}) > 0$ . Since  $g$  is convex, a mean preserving spread in the distribution of  $\tilde{R}$  increases the right hand side of the optimality condition

$$u'(Y_0 - s^*) = \beta E[u'(s^* \tilde{R}) \tilde{R}]$$

and  $s^*$  must increase to maintain the equality. The precautionary saving effect dominates if the coefficient of relative prudence exceeds 2.

## Risk Aversion and Saving Behavior

Conversely, if  $2 > P_R(Y)$ , then  $g''(\tilde{R}) < 0$ . Since  $g$  is concave, a mean preserving spread in the distribution of  $\tilde{R}$  decreases the right hand side of the optimality condition

$$u'(Y_0 - s^*) = \beta E[u'(s^* \tilde{R}) \tilde{R}]$$

and  $s^*$  must decrease to maintain the equality. The negative effect of risk on saving dominates if the coefficient of relative prudence is less than 2.

## Risk Aversion and Saving Behavior

To apply these results, let's calculate the coefficient of relative prudence implied by the CRRA utility function

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma},$$

with  $\gamma > 0$ . Since  $u'(Y) = Y^{-\gamma}$ ,

$$u''(Y) = -\gamma Y^{-\gamma-1} \text{ and } u'''(Y) = \gamma(\gamma + 1)Y^{-\gamma-2}$$

imply

$$P_R(Y) = -\frac{Y u'''(Y)}{u''(Y)} = \frac{Y \gamma(\gamma + 1) Y^{-\gamma-2}}{\gamma Y^{-\gamma-1}} = \gamma + 1.$$

## Risk Aversion and Saving Behavior

Hence, the CRRA utility function

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

implies both a constant coefficient of relative risk aversion equal to  $\gamma$  and a constant coefficient of relative prudence equal to  $\gamma + 1$ .

If  $\gamma > 1$ , saving rises in response to a mean preserving spread in the distribution of  $\tilde{R}$ . When  $\gamma < 1$ , saving falls. In the special case  $\gamma = 1$  of logarithmic utility, saving is unaffected.

## Separating Risk and Time Preferences

Our first set of results focused on the optimal choice of  $a$ , the amount of wealth to allocate to a risky asset.

Our second set of results focused on the optimal choice of  $s$ , the amount of saving to carry from period  $t = 0$  to period  $t = 1$ .

Now let's combine the two problems to consider the simultaneous choices of  $a$  and  $s$ .

## Separating Risk and Time Preferences

Suppose again that there are two periods,  $t = 0$  and  $t = 1$ , and let

$Y_0 =$  initial wealth

$s =$  amount saved in period  $t = 0$

$c_0 = Y_0 - s =$  amount consumed in period  $t = 0$

$a =$  amount allocated to stocks in period  $t = 0$

$s - a =$  amount allocated to the riskless asset in period  $t = 0$

$\tilde{r} =$  random return on stocks

$r_f =$  return on riskless asset

$c_1 =$  amount consumed in period  $t = 1$

## Separating Risk and Time Preferences

$Y_0$  = initial wealth

$s$  = amount saved in period  $t = 0$

$c_0 = Y_0 - s$  = amount consumed in period  $t = 0$

$a$  = amount allocated to stocks in period  $t = 0$

$s - a$  = amount allocated to the riskless asset in period  $t = 0$

$\tilde{r}$  = random return on stocks

$r_f$  = return on riskless asset

$c_1$  = amount consumed in period  $t = 1$

Then

$$c_1 = (1 + r_f)(s - a) + a(1 + \tilde{r}) = (1 + r_f)s + a(\tilde{r} - r_f).$$

## Separating Risk and Time Preferences

If the investor again has vN-M expected utility

$$u(c_0) + \beta E[u(c_1)] = u(Y_0 - s) + \beta E\{u[s(1 + r_f) + a(\tilde{r} - r)]\}$$

his or her problem can be stated as

$$\max_{s,a} u(Y_0 - s) + \beta E\{u[s(1 + r_f) + a(\tilde{r} - r)]\}$$



## Separating Risk and Time Preferences

$$\max_{s,a} u(Y_0 - s) + \beta E\{u[s(1 + r_f) + a(\tilde{r} - r)]\}$$

The first-order condition for  $s$  is

$$u'(Y_0 - s^*) = \beta(1 + r_f)E\{u'[s^*(1 + r_f) + a^*(\tilde{r} - r)]\}$$

and the first-order condition for  $a$  is

$$\beta E\{u'[s^*(1 + r_f) + a^*(\tilde{r} - r)](\tilde{r} - r_f)\} = 0$$

## Separating Risk and Time Preferences

The first-order conditions

$$u'(Y_0 - s^*) = \beta(1 + r_f)E\{u'[s^*(1 + r_f) + a^*(\tilde{r} - r)]\}$$

$$\beta E\{u'[s^*(1 + r_f) + a^*(\tilde{r} - r)](\tilde{r} - r_f)\} = 0$$

form a system of two equations in the two unknowns  $a^*$  and  $s^*$ , which can be solved numerically using a computer.

The model can be enriched further by considering additional periods  $t = 0, 1, 2, \dots, T$  and introducing labor income.

## Separating Risk and Time Preferences

Note, however, that the first-order condition for  $a$

$$\beta E\{u'[s^*(1 + r_f) + a^*(\tilde{r} - r)](\tilde{r} - r_f)\} = 0$$

takes the same form as in the simpler problem without saving:

$$E\{u'[Y_0(1 + r_f) + a^*(\tilde{r} - r_f)](\tilde{r} - r_f)\} = 0.$$

Hence, some of our previous results carry over to the more general case. With CRRA utility, for example,  $a^*$  will change proportionally with  $s^*$ , to maintain an optimal fraction of saving allocated to the risky asset.

## Separating Risk and Time Preferences

As a final exercise, let's return to the optimal saving problem

$$\max_s u(Y_0 - s) + \beta E[u(s\tilde{R})],$$

but simplify by eliminating randomness from the return  $\tilde{R}$  and by assuming from the start that the utility function takes the CRRA form:

$$\max_s \frac{(Y_0 - s)^{1-\gamma} - 1}{1 - \gamma} + \beta \left[ \frac{(sR)^{1-\gamma} - 1}{1 - \gamma} \right]$$

## Separating Risk and Time Preferences

$$\max_s \frac{(Y_0 - s)^{1-\gamma} - 1}{1-\gamma} + \beta \left[ \frac{(sR)^{1-\gamma} - 1}{1-\gamma} \right]$$

The first-order condition for the optimal choice of  $s$  is

$$(Y_0 - s)^{-\gamma} = \beta(sR)^{-\gamma} R$$

or, recalling that  $c_0 = Y_0 - s$  and  $c_1 = sR$ ,

$$c_0^{-\gamma} = \beta R c_1^{-\gamma}$$

## Separating Risk and Time Preferences

$$c_0^{-\gamma} = \beta R c_1^{-\gamma}$$

$$(c_1/c_0)^{\gamma} = \beta R$$

$$c_1/c_0 = (\beta R)^{1/\gamma}$$

$$\ln(c_1/c_0) = (1/\gamma) \ln(\beta) + (1/\gamma) \ln(R)$$

This last expression reveals that with this preference specification,  $\gamma$  measures the constant coefficient of relative risk aversion, but

$$\frac{1}{\gamma} = \frac{d \ln(c_1/c_0)}{d \ln(R)}$$

measures the constant **elasticity of intertemporal substitution**.

## Separating Risk and Time Preferences

Although the link between aversion to risk ( $\gamma$ ) and willingness to substitute consumption intertemporally ( $1/\gamma$ ) is particularly clear in the CRRA case, it holds more generally, since both features of preferences are reflected in the concavity of the Bernoulli utility function in the vN-M expected utility framework.

## Separating Risk and Time Preferences

In response to evidence that this link between risk aversion and intertemporal substitution is too restrictive to describe optimal saving and investment behavior, a more general preference specification is proposed by Larry Epstein and Stanley Zin, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica* Vol.57 (July 1989): 00.937-969.



## Separating Risk and Time Preferences

Although Epstein and Zin work in a multi-period framework, a simple two-period version of their proposed utility function over consumption  $c_0$  at  $t = 0$  and consumption  $\tilde{c}_1$ , possibly dependent on random asset returns, at  $t = 1$ , is

$$U(c_0, \tilde{c}_1) = \left\{ (1 - \beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[ (E(\tilde{c}_1^{1-\alpha}))^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

## Separating Risk and Time Preferences

$$U(c_0, \tilde{c}_1) = \left\{ (1 - \beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[ (E(\tilde{c}_1^{1-\alpha}))^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

Note, first, that if there is no uncertainty, so that  $\tilde{c}_1 = c_1$  and  $E(\tilde{c}_1)^{1-\alpha} = c_1^{1-\alpha}$ , then this utility function implies

$$\begin{aligned} U(c_0, c_1) &= \left\{ (1 - \beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[ (c_1^{1-\alpha})^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}} \\ &= \left\{ (1 - \beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta c_1^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

## Separating Risk and Time Preferences

Without uncertainty,

$$U(c_0, c_1) = \left\{ (1 - \beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta c_1^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

Define

$$V(c_0, c_1) = [U(c_0, c_1)]^{\frac{\sigma-1}{\sigma}} = (1 - \beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta c_1^{\frac{\sigma-1}{\sigma}}$$

and note that

$$\frac{\sigma - 1}{\sigma} = 1 - \frac{1}{\sigma}$$

to see that under certainty, the Epstein-Zin utility function implies an elasticity of intertemporal substitution equal to  $\sigma$ .

## Separating Risk and Time Preferences

$$U(c_0, \tilde{c}_1) = \left\{ (1 - \beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[ (E(\tilde{c}_1^{1-\alpha}))^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

On the other hand, under uncertainty, once period  $t = 1$  arrives, the investor cares about

$$E(\tilde{c}_1^{1-\alpha})$$

so  $\alpha$  is like the coefficient of relative risk aversion. Hence, the Epstein-Zin utility function allows the coefficient of relative risk aversion  $\alpha$  to differ from the inverse of the elasticity of intertemporal substitution  $\sigma$ .

## Separating Risk and Time Preferences

Note that under uncertainty, when  $\alpha = 1/\sigma$ ,

$$1 - \alpha = \frac{\sigma - 1}{\sigma}$$

and

$$\begin{aligned} U(c_0, \tilde{c}_1) &= \left\{ (1 - \beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[ (E(\tilde{c}_1^{1-\alpha}))^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}} \\ &= \left\{ (1 - \beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta E(\tilde{c}_1^{\frac{\sigma-1}{\sigma}}) \right\}^{\frac{\sigma}{\sigma-1}} \\ &= \left\{ (1 - \beta)c_0^{1-\alpha} + \beta E(\tilde{c}_1^{1-\alpha}) \right\}^{\frac{1}{1-\alpha}} \end{aligned}$$

## Separating Risk and Time Preferences

Under uncertainty, when  $\alpha = 1/\sigma$ ,

$$U(c_0, \tilde{c}_1) = \left\{ (1 - \beta)c_0^{1-\alpha} + \beta E(\tilde{c}_1^{1-\alpha}) \right\}^{\frac{1}{1-\alpha}}$$

Define

$$V(c_0, \tilde{c}_1) = [U(c_0, \tilde{c}_1)]^{1-\alpha} = (1 - \beta)c_0^{1-\alpha} + \beta E(\tilde{c}_1^{1-\alpha})$$

to see that in this case, the Epstein-Zin specification collapses to the standard CRRA case, where  $\alpha$  measures the coefficient of relative risk aversion and  $1/\alpha$  measures the elasticity of intertemporal substitution.

## Separating Risk and Time Preferences

Finally, note that in the general Epstein-Zin formulation

$$U(c_0, \tilde{c}_1) = \left\{ (1 - \beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[ (E(\tilde{c}_1^{1-\alpha}))^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

The expectation  $E(\tilde{c}_1^{1-\alpha})$  gets raised to the power

$$\left( \frac{1}{1-\alpha} \right) \left( \frac{\sigma-1}{\sigma} \right)$$

Unless  $\alpha = 1/\sigma$ , so that this product equals one, the probabilities of different states at  $t = 1$  will enter this utility function nonlinearly: the Epstein-Zin nonexpected utility function is a special case of those considered earlier by Kreps and Porteus.

## Separating Risk and Time Preferences

Hence, Epstein and Zin show that the coefficient of relative risk aversion and the elasticity of intertemporal substitution can be disentangled, but only at the cost of departing from the vN-M expected utility framework.

Alternatively, we can think of Epstein and Zin's study as giving us another reason to be interested in nonexpected utility: besides describing preferences over early versus late resolution of uncertainty, it also allows risk and time preferences to be separated.