

ECON 337901

FINANCIAL ECONOMICS

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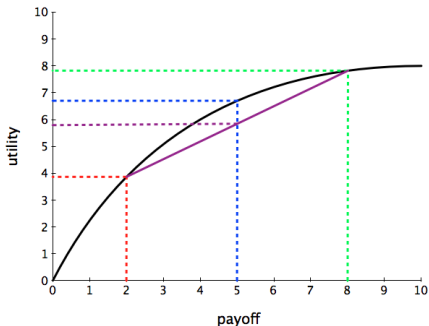
4 Measuring Risk and Risk Aversion

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Measuring Risk Aversion

We've already seen that within the von Neumann-Morgenstern expected utility framework, risk aversion enters through the concavity of the Bernoulli utility function.

Expected Utility Functions



When u is concave, a payoff of 5 for sure is preferred to a payoff of 8 with probability $1/2$ and 2 with probability $1/2$.

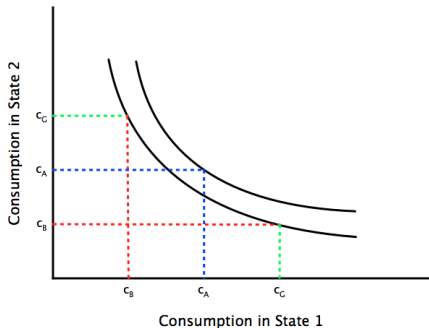
Measuring Risk Aversion

We've also seen previously that concavity of the utility function is related to convexity of indifference curves.

In standard microeconomic theory, this feature of preferences represents a “taste for diversity.”

Under uncertainty, it represents a desire to smooth consumption across future states of the world.

Expected Utility Functions



A risk averse consumer prefers $c_A = (c_G + c_B)/2$ in both states to c_G in one state and c_B in the other.

Measuring Risk Aversion

Mathematically, $u'(p) > 0$ means that an investor prefers higher payoffs to lower payoffs, and $u''(p) < 0$ means that the investor is risk averse.

But is there a way of quantifying an investor's degree of risk aversion?

And is there a criterion according to which we might judge one investor to be more risk averse than another?

Measuring Risk Aversion

Since $u''(p) < 0$ makes an investor risk averse, one conjecture would be to say that an investor with Bernoulli utility function $v(p)$ is more risk averse than another investor with Bernoulli utility function $u(p)$ if $v''(p) < u''(p)$ for all payoffs p .

But does a “more concave” Bernoulli utility function **always** correspond to greater risk aversion?

Unfortunately, no.

Measuring Risk Aversion

Recall that the preference ordering of an investor with vN-M utility function

$$U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y)$$

is also represented by the vN-M utility function

$$V(x, y, \pi) = \alpha U(x, y, \pi)$$

for **any** value of $\alpha > 0$.

Measuring Risk Aversion

But with

$$\begin{aligned}V(x, y, \pi) &= \alpha U(x, y, \pi) \\ &= \alpha \pi u(x) + \alpha(1 - \pi)u(y) \\ &= \pi \alpha u(x) + (1 - \pi) \alpha u(y) \\ &= \pi v(x) + (1 - \pi)v(y)\end{aligned}$$

where

$$v(p) = \alpha u(p)$$

for any payoff p .

Measuring Risk Aversion

Now

$$v(p) = \alpha u(p),$$

implies

$$v'(p) = \alpha u'(p)$$

and

$$v''(p) = \alpha u''(p),$$

By making α larger or smaller, the Bernoulli utility function can be made “more” or “less” concave without changing the underlying preference ordering.

Measuring Risk Aversion

In the mid-1960s, Kenneth Arrow and John Pratt proposed two alternative measures of risk aversion that are immune to this problem:

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Y u''(Y)}{u'(Y)} = \text{coefficient of relative risk aversion}$$

where Y measures the investor's income level.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Y u''(Y)}{u'(Y)} = \text{coefficient of relative risk aversion}$$

Absolute risk aversion applies to bets over absolute dollar amounts: $\pm \$1000$.

Relative risk aversion applies to bets expressed relative to (as a fraction of) income: ± 1 percent of Y .

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Y u''(Y)}{u'(Y)} = \text{coefficient of relative risk aversion}$$

Since $v(p) = \alpha u(p)$ implies $v'(p) = \alpha u'(p)$ and $v''(p) = \alpha u''(p)$, these measures are invariant to affine transformations of the Bernoulli utility function.

Measuring Risk Aversion

Two alternative measures of risk aversion are

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{coefficient of relative risk aversion}$$

where Y measures the investor's income level.

And since both measures have a minus sign out in front, both are positive and increase when risk aversion rises.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Y u''(Y)}{u'(Y)} = \text{coefficient of relative risk aversion}$$

The notation $R_A(Y)$ and $R_R(Y)$ emphasizes that both measures of risk aversion can depend on the investor's income Y . A given bet can seem more or less risky, depending on the investor's income.

Interpreting the Measures of Risk Aversion

To interpret the two measures of risk aversion, it is helpful to recall from calculus the theorem stated by Brook Taylor (England, 1685-1731), regarding the approximation of a function f using its derivatives: the “first-order” approximation

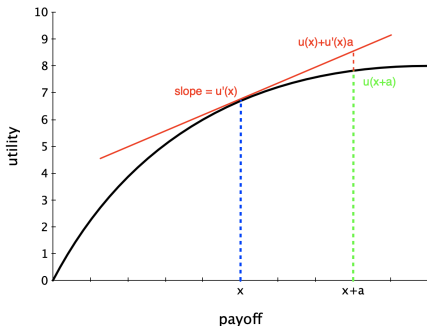
$$f(x + a) \approx f(x) + f'(x)a$$

and the “second-order” approximation

$$f(x + a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2.$$

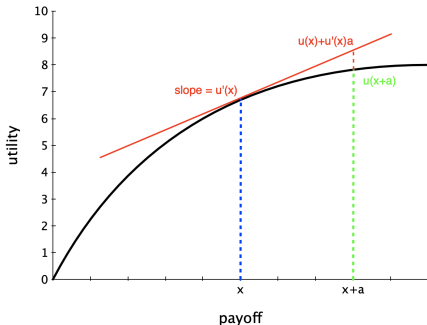
The second-order approximation is more accurate than the first, and both become more accurate as a becomes smaller.

Interpreting the Measures of Risk Aversion



The first-order (linear) approximation $u(x + a) \approx u(x) + u'(x)a$ overstates $u(x + a)$ when u is concave.

Interpreting the Measures of Risk Aversion



Since $u''(x) < 0$, the second-order (quadratic) approximation $u(x+a) \approx u(x) + u'(x)a + (1/2)u''(x)a^2$ will be more accurate.

Interpreting the Measures of Risk Aversion

Focusing first on the measure of absolute risk aversion, consider an investor with initial income Y who is offered a bet: win h with probability π and lose h with probability $1 - \pi$.

A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

Let π^* be the probability that is just high enough to get the investor to accept the bet.

Then π^* must satisfy

$$u(Y) = \pi^* u(Y + h) + (1 - \pi^*) u(Y - h).$$

Interpreting the Measures of Risk Aversion

Take second-order Taylor approximations to $u(Y + h)$ and $u(Y - h)$:

$$u(Y + h) \approx u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$u(Y - h) \approx u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) = \pi^* u(Y + h) + (1 - \pi^*) u(Y - h)$$

$$u(Y + h) \approx u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$u(Y - h) \approx u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2$$

imply

$$\begin{aligned} u(Y) &\approx \pi^* \left[u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2 \right] \\ &\quad + (1 - \pi^*) \left[u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2 \right] \end{aligned}$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx \pi^* \left[u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2 \right] \\ + (1 - \pi^*) \left[u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2 \right]$$

implies

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$0 \approx (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$0 \approx (2\pi^* - 1)u'(Y) + \frac{1}{2}u''(Y)h$$

$$2\pi^* u'(Y) \approx u'(Y) - \frac{1}{2}u''(Y)h$$

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} \left[-\frac{u''(Y)}{u'(Y)} \right] h$$

Interpreting the Measures of Risk Aversion

Since

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{coefficient of absolute risk aversion,}$$

it follows from these calculations that

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} \left[-\frac{u''(Y)}{u'(Y)} \right] h = \frac{1}{2} + \frac{1}{4} h R_A(Y) > \frac{1}{2}.$$

The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus h relates directly to the coefficient of absolute risk aversion.

Interpreting the Measures of Risk Aversion

As an example, suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus $h = \$1000$?

And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $h = \$1000$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}hR_A(Y)$$

implies

$$0.75 \approx 0.50 + \frac{1000}{4}R_A(Y)$$

$$0.25 \approx 250R_A(Y)$$

$$R_A(Y) \approx \frac{0.25}{250} = 0.001.$$

Interpreting the Measures of Risk Aversion

Realistically, a bet over \$1000 is probably going to seem more risky to someone who starts out with less income.

In general, $R_A(Y)$ can depend on Y . More specifically, it seems likely that $R_A(Y)$ decreases when Y goes up, so that

$$R'_A(Y) < 0.$$

Interpreting the Measures of Risk Aversion

Suppose, however, that the Bernoulli utility function takes the form

$$u(Y) = -\frac{1}{\nu}e^{-\nu Y},$$

where $\nu > 0$ and e^x is the exponential function ($e \approx 2.718$).

Recall that exponential function has the special property that

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

and by the chain rule

$$g(x) = e^{\alpha x} \Rightarrow g'(x) = \alpha e^{\alpha x}$$

Interpreting the Measures of Risk Aversion

With

$$u(Y) = -\frac{1}{\nu}e^{-\nu Y},$$

it follows that

$$u'(Y) = -\frac{1}{\nu}e^{-\nu Y}(-\nu) = e^{-\nu Y}$$

$$u''(Y) = -\nu e^{-\nu Y}$$

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \frac{\nu e^{-\nu Y}}{e^{-\nu Y}} = \nu,$$

so that this utility function displays **constant absolute risk aversion**, which does not depend on income.

Interpreting the Measures of Risk Aversion

So if we were willing to make the assumption of constant absolute risk aversion, we could use the results from our example, where an investor requires $\pi^* = 0.75$ to accept a bet with $h = \$1000$ to set $\nu = 0.001$ in

$$u(Y) = -\frac{1}{\nu}e^{-\nu Y},$$

and thereby tailor portfolio decisions specifically for this investor.

Interpreting the Measures of Risk Aversion

Absolute risk aversion describes an investor's attitude towards **absolute** bets of plus or minus h .

A similar analysis shows that relative risk aversion describes an investor's attitude towards **relative** bets of plus or minus kY , so that now, k is a fraction of total income.

Interpreting the Measures of Risk Aversion

Consider an investor with initial income Y who is offered a bet: win kY with probability π and lose kY with probability $1 - \pi$.

A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

Let π^* be the probability that is just high enough to get the investor to accept the bet.

Now π^* must satisfy

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*) u(Y - Yk).$$

Interpreting the Measures of Risk Aversion

Take second-order Taylor approximations to $u(Y + Yk)$ and $u(Y - Yk)$:

$$u(Y + Yk) \approx u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$u(Y - Yk) \approx u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*) u(Y - Yk)$$

$$u(Y + Yk) \approx u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$u(Y - Yk) \approx u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

imply

$$\begin{aligned} u(Y) \approx & \pi^* \left[u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2 \right] \\ & + (1 - \pi^*) \left[u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2 \right] \end{aligned}$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx \pi^* \left[u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2 \right] \\ + (1 - \pi^*) \left[u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2 \right]$$

implies

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$0 \approx (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$0 \approx (2\pi^* - 1)u'(Y) + \frac{1}{2}u''(Y)Yk$$

$$2\pi^* u'(Y) \approx u'(Y) - \frac{1}{2}u''(Y)Yk$$

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} \left[-\frac{Yu''(Y)}{u'(Y)} \right] k$$

Interpreting the Measures of Risk Aversion

Since

$$R_R(Y) = -\frac{Y u''(Y)}{u'(Y)} = \text{coefficient of relative risk aversion,}$$

it follows from these calculations that

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} \left[-\frac{Y u''(Y)}{u'(Y)} \right] k = \frac{1}{2} + \frac{1}{4} k R_R(Y) > \frac{1}{2}.$$

The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus kY relates directly to the coefficient of relative risk aversion.

Interpreting the Measures of Risk Aversion

Suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus one percent ($k = 0.01$) of your income?

And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $k = 0.01$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}kR_R(Y)$$

implies

$$0.75 \approx 0.50 + \frac{0.01}{4}R_R(Y)$$

$$0.25 \approx 0.0025R_R(Y)$$

$$R_R(Y) \approx \frac{0.25}{0.0025} = 100.$$

Interpreting the Measures of Risk Aversion

Again, our notation $R_R(Y)$ allows relative risk aversion to depend on income Y .

On the other hand, since the coefficient of relative risk aversion describes aversion to risk over bets that are expressed relative to income, it is more plausible to assume that investors have constant relative risk aversion.

Interpreting the Measures of Risk Aversion

Suppose, therefore, that the Bernoulli utility function takes the form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

where $\gamma > 0$. For this function, Guillaume de l'Hôpital's (France, 1661-1704) rule implies that when $\gamma = 1$

$$\frac{Y^{1-\gamma} - 1}{1 - \gamma} = \ln(Y),$$

where \ln denotes the natural logarithm. This was the form that Daniel Bernoulli used to describe preferences over payoffs.

Interpreting the Measures of Risk Aversion

To see this, consider $Y^{1-\gamma}$, not as a function of Y but as a function of γ ,

$$f(\gamma) = Y^{1-\gamma},$$

and take the natural logarithm of both sides to obtain

$$\ln(f(\gamma)) = (1 - \gamma) \ln(Y).$$

Recall, also, that if $g(x) = \ln(x)$,

$$g'(x) = \frac{d}{dx} \ln(x) = \frac{1}{x}.$$

Interpreting the Measures of Risk Aversion

Hence, the chain rule implies that if

$$\ln(f(\gamma)) = (1 - \gamma) \ln(Y),$$

then

$$\frac{d}{d\gamma} \ln(f(\gamma)) = \frac{f'(\gamma)}{f(\gamma)} = -\ln(Y)$$

and hence

$$f'(\gamma) = -\ln(Y)f(\gamma) = -\ln(Y)Y^{1-\gamma}$$

Interpreting the Measures of Risk Aversion

Hence, by l'Hôpital's rule

$$\begin{aligned}\lim_{\gamma \rightarrow 1} \frac{Y^{1-\gamma} - 1}{1 - \gamma} &= \lim_{\gamma \rightarrow 1} \frac{\frac{d}{d\gamma}(Y^{1-\gamma} - 1)}{\frac{d}{d\gamma}(1 - \gamma)} \\ &= \lim_{\gamma \rightarrow 1} \frac{-\ln(Y)Y^{1-\gamma}}{-1} \\ &= \ln(Y).\end{aligned}$$

Interpreting the Measures of Risk Aversion

With

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

it follows that

$$u'(Y) = Y^{-\gamma}$$

$$u''(Y) = -\gamma Y^{-\gamma-1}$$

$$R_R(Y) = -\frac{Y u''(Y)}{u'(Y)} = \frac{Y \gamma Y^{-\gamma-1}}{Y^{-\gamma}} = \gamma,$$

so that this utility function displays **constant relative risk aversion**, which does not depend on income.

Interpreting the Measures of Risk Aversion

So if we were willing to make the assumption of constant relative risk aversion, we could use the results from our example, where an investor requires $\pi^* = 0.75$ to accept a bet with $k = 0.01$ to set $\gamma = 100$ in

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

and thereby tailor portfolio decisions specifically for this investor.

Interpreting the Measures of Risk Aversion

Finally, suppose that we do away with the concavity of the Bernoulli utility function and simply assume that

$$u(p) = \alpha p + \beta,$$

where $\alpha > 0$, so that higher payoffs are preferred to lower payoffs. For this utility function,

$$u'(Y) = \alpha \text{ and } u''(Y) = 0$$

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = 0 \text{ and } R_R(Y) = -\frac{Y u''(Y)}{u'(Y)} = 0.$$

This investor is **risk-neutral** and cares only about expected payoffs.

Risk Premium and Certainty Equivalent

Our thought experiments so far have asked about how probabilities need to be boosted in order to induce a risk-averse investor to accept an absolute or relative bet.

Let's take step away from gambling and towards investing by asking: suppose that an investor with income Y has the opportunity to buy an asset with a payoff \tilde{Z} that is random and has expected value $E(\tilde{Z})$.

Risk Premium and Certainty Equivalent

If this investor is risk-averse and has vN-M expected utility, he or she will always prefer an alternative asset that pays off $E(\tilde{Z})$ for sure. Mathematically,

$$u[Y + E(\tilde{Z})] \geq E[u(Y + \tilde{Z})],$$

“the utility of the expectation is greater than the expectation of utility.”

Risk Premium and Certainty Equivalent

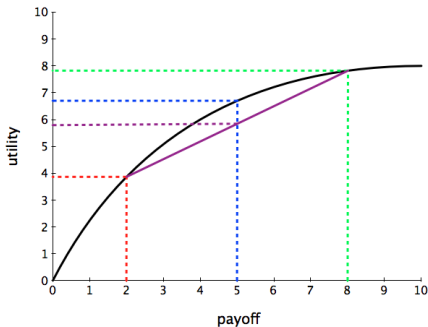
This follows from a result proven by Johan Jensen (Denmark, 1859-1925).

Theorem (Jensen's Inequality) Let g be a concave function and \tilde{x} be a random variable. Then

$$g[E(\tilde{x})] \geq E[g(\tilde{x})].$$

Furthermore, if g is strictly concave and the probability that $\tilde{x} \neq E(\tilde{x})$ is greater than zero, the inequality is strict.

Risk Premium and Certainty Equivalent



This graph illustrates a special case of Jensen's inequality. The result holds much more generally.

Risk Premium and Certainty Equivalent

An implication of Jensen's inequality is that the **maximum** riskless payoff that a risk-averse investor is willing to exchange for the asset with random payoff \tilde{Z} , called the **certainty equivalent** for that asset, will always be less than $E(\tilde{Z})$.

Since

$$u[Y + E(\tilde{Z})] \geq E[u(Y + \tilde{Z})],$$

the certainty equivalent $CE(\tilde{Z})$ defined by

$$u[Y + CE(\tilde{Z})] = E[u(Y + \tilde{Z})]$$

also satisfies $CE(\tilde{Z}) \leq E(\tilde{Z})$.

Risk Premium and Certainty Equivalent

Since

$$u[Y + E(\tilde{Z})] \geq E[u(Y + \tilde{Z})],$$

the certainty equivalent $CE(\tilde{Z})$ defined by

$$u[Y + CE(\tilde{Z})] = E[u(Y + \tilde{Z})]$$

also satisfies $CE(\tilde{Z}) \leq E(\tilde{Z})$.

The difference between the higher expected value $E(\tilde{Z})$ and the smaller certainty equivalent $CE(\tilde{Z})$ can then be used to define the positive **risk premium** $\Psi(\tilde{Z})$ for the asset:

$$\Psi(\tilde{Z}) = E(\tilde{Z}) - CE(\tilde{Z}) \geq 0.$$

Risk Premium and Certainty Equivalent

The certainty equivalent and risk premium are “two sides of the same coin”

$$\psi(\tilde{Z}) = E(\tilde{Z}) - CE(\tilde{Z})$$

$CE(\tilde{Z})$ = lesser amount the investor is willing to accept to remain invested in the risk-free asset

$\psi(\tilde{Z})$ = extra amount the investor needs to take on additional risk

Risk Premium and Certainty Equivalent

Combining the definitions of the certainty equivalent $CE(\tilde{Z})$,

$$E[u(Y + \tilde{Z})] = u[Y + CE(\tilde{Z})],$$

and the risk premium $\Psi(\tilde{Z})$,

$$CE(\tilde{Z}) = E(\tilde{Z}) - \Psi(\tilde{Z}),$$

yields

$$E[u(Y + \tilde{Z})] = u[Y + E(\tilde{Z}) - \Psi(\tilde{Z})],$$

which we can use to link the risk premium $\Psi(\tilde{Z})$ to our measures of risk aversion.

Risk Premium and Certainty Equivalent

Recall that if the random variable X can take on n possible values, X_1, X_2, \dots, X_n , with probabilities $\pi_1, \pi_2, \dots, \pi_n$, then the expected value of X is defined as

$$E(X) = \pi_1 X_1 + \pi_2 X_2 + \dots + \pi_n X_n.$$

It follows from this definition that the random variable defined by αX , with $\alpha \in \mathbb{R}$, is such that

$$\begin{aligned} E(\alpha X) &= \pi_1 \alpha X_1 + \pi_2 \alpha X_2 + \dots + \pi_n \alpha X_n \\ &= \alpha (\pi_1 X_1 + \pi_2 X_2 + \dots + \pi_n X_n) = \alpha E(X). \end{aligned}$$

Risk Premium and Certainty Equivalent

With this fact in mind, return to

$$E[u(Y + \tilde{Z})] = u[Y + E(\tilde{Z}) - \Psi(\tilde{Z})],$$

but let

$$\begin{aligned} Y^* &= Y + E(\tilde{Z}) \\ &= \text{income plus expected payout from the risky asset} \end{aligned}$$

so that

$$E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} = u[Y^* - \Psi(\tilde{Z})].$$

Risk Premium and Certainty Equivalent

Take a second-order Taylor approximation to $u[Y^* + \tilde{Z} - E(\tilde{Z})]$, viewing $\tilde{Z} - E(\tilde{Z})$ as the “size of the bet”

$$u[Y^* + \tilde{Z} - E(\tilde{Z})] \approx u(Y^*) + u'(Y^*)[\tilde{Z} - E(\tilde{Z})] + \frac{1}{2}u''(Y^*)[\tilde{Z} - E(\tilde{Z})]^2.$$

Risk Premium and Certainty Equivalent

Now take the expected value on both sides and simplify, using the fact that Y^* is not random:

$$\begin{aligned} E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} &\approx E[u(Y^*)] \\ &\quad + E\{u'(Y^*)[\tilde{Z} - E(\tilde{Z})]\} \\ &\quad + E\left\{\frac{1}{2}u''(Y^*)[\tilde{Z} - E(\tilde{Z})]^2\right\} \\ &= u(Y^*) + u'(Y^*)E[\tilde{Z} - E(\tilde{Z})] \\ &\quad + \frac{1}{2}u''(Y^*)E\{[\tilde{Z} - E(\tilde{Z})]^2\}. \end{aligned}$$

Risk Premium and Certainty Equivalent

Finally, use the fact that $E[\tilde{Z} - E(\tilde{Z})] = 0$ and the definition of the variance of \tilde{Z} to simplify further:

$$\begin{aligned} E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} &\approx u(Y^*) + u'(Y^*)E[\tilde{Z} - E(\tilde{Z})] \\ &\quad + \frac{1}{2}u''(Y^*)E\{[\tilde{Z} - E(\tilde{Z})]^2\} \\ &= u(Y^*) + \frac{1}{2}\sigma^2(\tilde{Z})u''(Y^*). \end{aligned}$$

Risk Premium and Certainty Equivalent

On the other side of our original equation, consider a first-order Taylor approximation to $u[Y^* - \Psi(\tilde{Z})]$:

$$u[Y^* - \Psi(\tilde{Z})] \approx u(Y^*) - u'(Y^*)\Psi(\tilde{Z}).$$

Risk Premium and Certainty Equivalent

Hence, the equation defining the risk premium

$$E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} = u[Y^* - \Psi(\tilde{Z})],$$

and the approximations

$$E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} \approx u(Y^*) + \frac{1}{2}\sigma^2(\tilde{Z})u''(Y^*)$$

$$u[Y^* - \Psi(\tilde{Z})] \approx u(Y^*) - u'(Y^*)\Psi(\tilde{Z})$$

imply

$$\frac{1}{2}\sigma^2(\tilde{Z})u''(Y^*) \approx -u'(Y^*)\Psi(\tilde{Z}).$$

Risk Premium and Certainty Equivalent

$$\frac{1}{2}\sigma^2(\tilde{Z})u''(Y^*) \approx -u'(Y^*)\Psi(\tilde{Z})$$

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z}) \left[-\frac{u''(Y^*)}{u'(Y^*)} \right]$$

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z})R_A(Y^*) = \frac{1}{2}\sigma^2(\tilde{Z})R_A(Y + E(\tilde{Z})),$$

indicating that the risk premium depends directly on the coefficient of absolute risk aversion and the absolute “size of the bet” $\sigma^2(\tilde{Z})$.

Risk Premium and Certainty Equivalent

As an example, consider an investor with income $Y = 50000$ and utility function of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

with $\gamma = 5$, who is considering buying an asset with random payoff \tilde{Z} that equals 2000 with probability 1/2 and 0 with probability 1/2. For this asset

$$E(\tilde{Z}) = (1/2)2000 + (1/2)0 = 1000$$

$$\sigma^2(\tilde{Z}) = (1/2)(2000 - 1000)^2 + (1/2)(0 - 1000)^2 = 1000^2.$$

Risk Premium and Certainty Equivalent

Our approximation formula

$$\psi(\tilde{Z}) \approx \frac{1}{2} \sigma^2(\tilde{Z}) R_A(Y + E(\tilde{Z}))$$

indicates that

$$\psi(\tilde{Z}) \approx \frac{1}{2} (1000)^2 \left(\frac{5}{51000} \right) = 49.02$$

since $R_A(Y) = R_R(Y)/Y$.

Risk Premium and Certainty Equivalent

Now go back to the original formula defining the risk premium,

$$E[u(Y + \tilde{Z})] = u[Y + E(\tilde{Z}) - \Psi(\tilde{Z})],$$

and plug in the numbers to see that in this case, the exact value of $\Psi(\tilde{Z})$ must satisfy

$$\begin{aligned} & (1/2) \left(\frac{52000^{-4} - 1}{-4} \right) + (1/2) \left(\frac{50000^{-4} - 1}{-4} \right) \\ &= \frac{(51000 - \Psi(\tilde{Z}))^{-4} - 1}{-4}. \end{aligned}$$

Risk Premium and Certainty Equivalent

$$\begin{aligned} & (1/2) \left(\frac{52000^{-4} - 1}{-4} \right) + (1/2) \left(\frac{50000^{-4} - 1}{-4} \right) \\ &= \frac{(51000 - \Psi(\tilde{Z}))^{-4} - 1}{-4}. \end{aligned}$$

$$\begin{aligned} (1/2)52000^{-4} + (1/2)50000^{-4} &= (51000 - \Psi(\tilde{Z}))^{-4} \\ [(1/2)52000^{-4} + (1/2)50000^{-4}]^{-1/4} &= 51000 - \Psi(\tilde{Z}) \\ \Psi(\tilde{Z}) &= 51000 - [(1/2)52000^{-4} + (1/2)50000^{-4}]^{-1/4} \\ \Psi(\tilde{Z}) &= 48.97. \end{aligned}$$

Risk Premium and Certainty Equivalent

The approximation $\Psi(\tilde{Z}) \approx 49.02$ or the exact solution $\Psi(\tilde{Z}) = 48.97$ imply that an investor with $Y = 50000$ and constant coefficient of relative risk aversion equal to 5 will give up a riskless payoff of up to about

$$CE(\tilde{Z}) = E(\tilde{Z}) - \Psi(\tilde{Z}) \approx 1000 - 49 = 951$$

for this risky asset with expected payoff equal to 1000.

Assessing the Level of Risk Aversion

We can use similar calculations to work through thought experiments that shed light on our own levels of risk aversion.

Suppose your income is $Y = 50000$ and you have the chance to buy an asset that pays 50000 with probability $1/2$ and 0 with probability $1/2$.

This asset has $E(\tilde{Z}) = (1/2)50000 + (1/2)0 = 25000$, but what is the maximum riskless payoff you would exchange for it?

Assessing the Level of Risk Aversion

Suppose your utility function is of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

and recall that the most you should pay for the asset is given by the certainty equivalent $CE(\tilde{Z})$ defined by

$$E[u(Y + \tilde{Z})] = u[Y + CE(\tilde{Z})].$$

Assessing the Level of Risk Aversion

$$E[u(Y + \tilde{Z})] = u[Y + CE(\tilde{Z})]$$

$$\begin{aligned} & (1/2) \left(\frac{100000^{1-\gamma} - 1}{1-\gamma} \right) + (1/2) \left(\frac{50000^{1-\gamma} - 1}{1-\gamma} \right) \\ = & \frac{(50000 + CE(\tilde{Z}))^{1-\gamma} - 1}{1-\gamma} \end{aligned}$$

$$CE(\tilde{Z}) = [(1/2)100000^{1-\gamma} + (1/2)50000^{1-\gamma}]^{1/(1-\gamma)} - 50000$$

Assessing the Level of Risk Aversion

Certainty equivalent for an asset that pays 50000 with probability 1/2 and 0 with probability 1/2 when income is 50000 and the coefficient of relative risk aversion is γ .

γ	$CE(\tilde{Z})$	$\psi(\tilde{Z})$	
0	25000	0	("risk neutrality," Pascal)
1	20711	4289	(log utility, D Bernoulli)
2	16667	8333	
3	13246	11754	
4	10571	14429	
5	8566	16434	
10	3991	21009	
20	1858	23142	
50	712	24288	

The Concept of Stochastic Dominance

It is important to recognize that the coefficients of absolute and relative risk aversion, $R_A(Y)$ and $R_R(Y)$, and the certainty equivalent $CE(\tilde{Z})$ and the risk premium $\Psi(\tilde{Z})$, all help describe or summarize investors' preferences over risky cash flows.

They do not directly represent differences in market or equilibrium prices or rates of return across riskless and risky assets.

The Concept of Stochastic Dominance

Since individuals will differ in their attitudes towards risk as in their preferences over everything else (“chacun à son goût,” as the French say), it is useful to ask whether there are properties of payoff distributions that will allow “preference-free” comparisons to be made across risky cash flows.

State-by-state dominance, as we’ve already seen, is one such property. But are there any others, which might be more widely applicable?

The Concept of Stochastic Dominance

Consider two assets, with random payoffs Z_1 and Z_2 :

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

There may be no state-by-state dominance, if the payoffs $Z_1 = 10$ and $Z_2 = 100$ can occur together and the payoffs $Z_1 = 100$ and $Z_2 = 10$ can occur together.

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

Because $E(Z_1) = 64$, $\sigma(Z_1) = 44$, $E(Z_2) = 244$, and $\sigma(Z_2) = 380$, there is no mean-variance dominance either.

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

But, intuitively, asset 2 “looks” better, because its distribution takes some of the probability of a payoff of 100 and “moves” that probability to the even higher payoff of 1000.

We can make this idea more concrete by looking at the distributions of these random payoffs in a different way.

The Concept of Stochastic Dominance

In probability theory, the **cumulative distribution function (cdf)** for a random variable X keeps track of the probability that the realized value of X will be less than or equal to x :

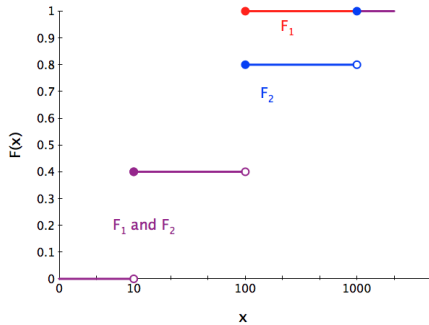
$$F(x) = \text{Prob}(X \leq x).$$

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

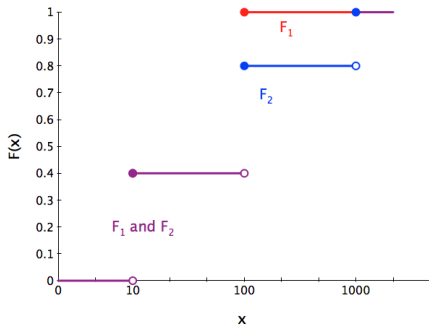
cdfs	$x < 10$	$10 \leq x < 100$	$100 \leq x < 1000$	$1000 \leq x$
$F_1(x)$	0.00	0.40	1.00	1.00
$F_2(x)$	0.00	0.40	0.80	1.00

The Concept of Stochastic Dominance



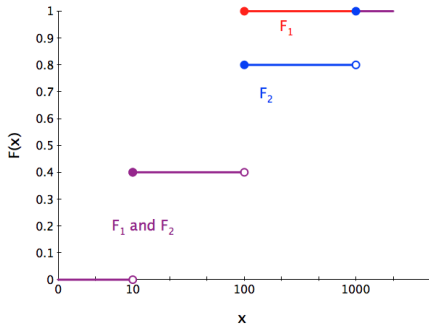
Cumulative distribution functions are always **nondecreasing**.

The Concept of Stochastic Dominance



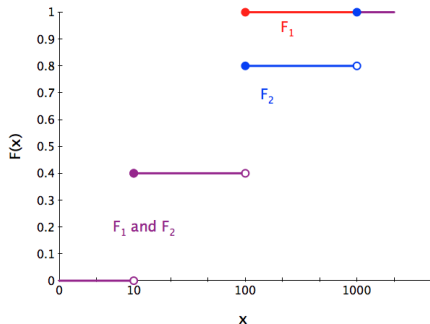
Cumulative distribution functions always satisfy $F(-\infty) = 0$, $F(\infty) = 1$ and $0 \leq F(x) \leq 1$.

The Concept of Stochastic Dominance



Cumulative distribution functions are always **càdlàg** (“continue à droite, limite à gauche”) or **RCLL** “right continuous with left limits.”

The Concept of Stochastic Dominance



The fact that $F_2(x)$ always lies below $F_1(x)$ formalizes the **first-order stochastic dominance** of Z_2 over Z_1 .

The Concept of Stochastic Dominance

cdfs	$x < 10$	$10 \leq x < 100$	$100 \leq x < 1000$	$1000 \leq x$
$F_1(x)$	0.00	0.40	1.00	1.00
$F_2(x)$	0.00	0.40	0.80	1.00

Asset 2 displays **first-order stochastic dominance** over asset 1 because $F_2(x) \leq F_1(x)$ for all possible values of x .

The Concept of Stochastic Dominance

Theorem Let $F_1(x)$ and $F_2(x)$ be the cumulative distribution functions for two assets with random payoffs Z_1 and Z_2 . Then

$$F_2(x) \leq F_1(x) \text{ for all } x,$$

that is, asset 2 displays first-order stochastic dominance over asset 1, if and only if

$$E[u(Z_2)] \geq E[u(Z_1)]$$

for **any** nondecreasing Bernoulli utility function u .

The Concept of Stochastic Dominance

First-order stochastic dominance is a weaker condition than state-by-state dominance, in that state-by-state dominance implies first-order stochastic dominance but first-order stochastic dominance does not necessarily imply state-by-state dominance.

But first-order stochastic dominance remains quite a strong condition. Since an asset that displays first-order stochastic dominance over all others will be preferred by **any** investor with vN-M utility who prefers higher payoffs to lower payoffs, the price of such an asset is likely to be bid up until the dominance goes away.

The Concept of Stochastic Dominance

Consider two more assets, with random payoffs Z_3 and Z_4 :

Payoffs	1	4	5	6	8	9
Probabilities for Z_3	0.33	0.00	0.00	0.33	0.33	0.00
Probabilities for Z_4	0.00	0.25	0.50	0.00	0.00	0.25

Asset 4 looks at least slightly better, since it always pays off at least 4 and has a non-trivial probability of 9. On the other hand, asset 3 has a higher probability of a payoff of 6 or more.

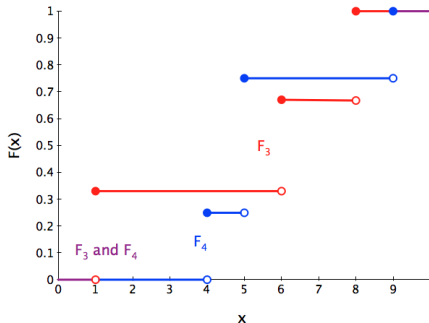
The Concept of Stochastic Dominance

Payoffs	1	4	5	6	8	9
Probabilities for Z_3	0.33	0.00	0.00	0.33	0.33	0.00
Probabilities for Z_4	0.00	0.25	0.50	0.00	0.00	0.25

cdfs	$x < 1$	$1 \leq x < 4$	$4 \leq x < 5$	$5 \leq x < 6$
$F_3(x)$	0.00	0.33	0.33	0.33
$F_4(x)$	0.00	0.00	0.25	0.75

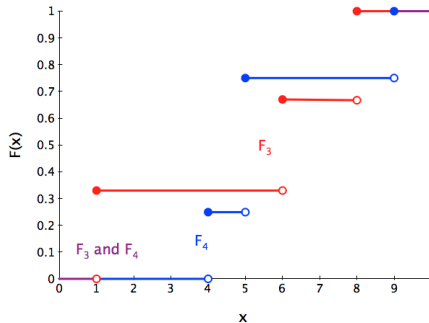
cdfs	$6 \leq x < 8$	$8 \leq x < 9$	$9 \leq x$
$F_3(x)$	0.66	1.00	1.00
$F_4(x)$	0.75	0.75	1.00

The Concept of Stochastic Dominance



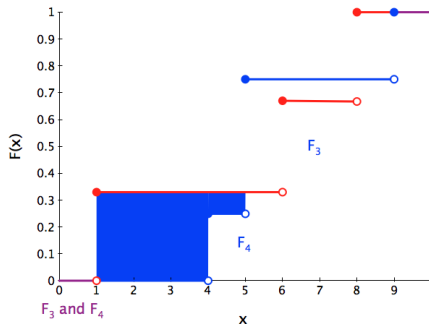
There is no first-order stochastic dominance, since $F_4(x) \leq F_3(x)$ for some values of x but $F_3(x) \leq F_4(x)$ for other values of x .

The Concept of Stochastic Dominance



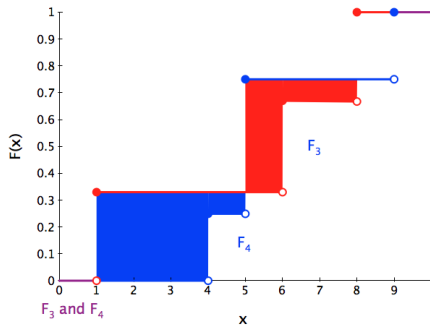
Still, as we move from left to right ...

The Concept of Stochastic Dominance



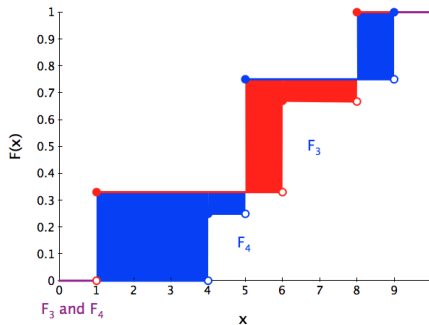
... the areas over which $F_4(x) \leq F_3(x)$...

The Concept of Stochastic Dominance



... seem to be consistently larger than the areas over which $F_3(x) \leq F_4(x)$.

The Concept of Stochastic Dominance



Thus, asset 4 displays **second-order stochastic dominance** over asset 3.

The Concept of Stochastic Dominance

cdfs	$x < 1$	$1 \leq x < 4$	$4 \leq x < 5$	$5 \leq x < 6$
$F_3(x)$	0.00	0.33	0.33	0.33
$F_4(x)$	0.00	0.00	0.25	0.75

cdfs	$6 \leq x < 8$	$8 \leq x < 9$	$9 \leq x$
$F_3(x)$	0.66	1.00	1.00
$F_4(x)$	0.75	0.75	1.00

Asset 4 displays **second-order stochastic dominance** over asset 4 since

$$\int_{-\infty}^{\bar{x}} [F_4(x) - F_3(x)] dx \leq 0$$

for all possible values of \bar{x} .

The Concept of Stochastic Dominance

Asset 4 displays **second-order stochastic dominance** over asset 4 since

$$\int_{-\infty}^{\bar{x}} [F_4(x) - F_3(x)] dx \leq 0$$

for all possible values of \bar{x} .

Note the integral in the definition runs from $-\infty$ up to \bar{x} , since we want to penalize assets with higher probabilities of lower payoffs (we moved from left to right in the graphs). Note, also, that the condition must hold for **all** values of \bar{x} .

The Concept of Stochastic Dominance

Theorem Let $F_3(x)$ and $F_4(x)$ be the cumulative distribution functions for two assets with random payoffs Z_3 and Z_4 . Then

$$\int_{-\infty}^{\bar{x}} [F_4(x) - F_3(x)] dx \leq 0 \text{ for all } \bar{x}$$

that is, asset 4 displays second-order stochastic dominance over asset 3, if and only if

$$E[u(Z_4)] \geq E[u(Z_3)]$$

for any nondecreasing **and concave** Bernoulli utility function u .

The Concept of Stochastic Dominance

Second-order stochastic dominance is a weaker condition than first-order stochastic dominance, in that first-order stochastic dominance implies second-order stochastic dominance but second-order stochastic dominance does not necessarily imply first-order stochastic dominance.

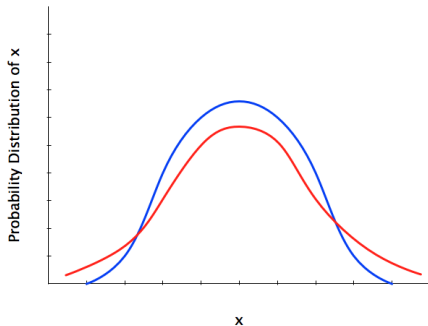
But second-order stochastic dominance remains a strong condition. Since an asset that displays second-order stochastic dominance over all others will be preferred by any **risk-averse** investor with $vN-M$ utility, the price of such an asset is likely to be bid up until the dominance goes away.

Mean Preserving Spreads

Comparisons based on state-by-state dominance, first-order stochastic dominance, and second-order stochastic dominance can reflect differences in the mean, or expected, payoff as well as in the standard deviation or variance of the payoff.

It is also useful, therefore, to consider an alternative criterion that focuses entirely on the standard deviation of a random payoff, as a measure of the riskiness of the corresponding asset, holding the mean or expected value fixed.

Mean Preserving Spreads



Graphically, a **mean preserving spread** takes probability from the center of a distribution and shifts it to the tails.

Mean Preserving Spreads

Mathematically, one way of producing a mean preserving spread is to take one random variable X_1 and define a second, X_2 , by adding “noise” in the form of a third, zero-mean random variable Z :

$$X_2 = X_1 + Z,$$

where $E(Z) = 0$.

Mean Preserving Spreads

As an example, suppose that

$$X_1 = \begin{cases} 5 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/2 \end{cases}$$

$$Z = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

then

$$X_2 = X_1 + Z = \begin{cases} 6 & \text{with probability } 1/4 \\ 4 & \text{with probability } 1/4 \\ 3 & \text{with probability } 1/4 \\ 1 & \text{with probability } 1/4 \end{cases}$$

$E(X_1) = E(X_2) = 3.5$, but if these are random payoffs, asset 2 seems riskier.

Mean Preserving Spreads

The following theorem relates the concept of a mean preserving spread to the previous concept of second-order stochastic dominance.

Theorem Let X_1 and X_2 , with $E(X_1) = E(X_2)$, be random payoffs on two assets. Then the following two statements are equivalent: (i) $X_2 = X_1 + Z$ for some random variable Z with $E(Z) = 0$ and (ii) asset 1 displays second-order stochastic dominance over asset 2.

Mean Preserving Spreads

Theorem Consider two assets with random payoffs Z_1 and Z_2 . Asset 2 displays second-order stochastic dominance over asset 1 if and only if

$$E[u(Z_2)] \geq E[u(Z_1)]$$

for any nondecreasing **and concave** Bernoulli utility function u .

This theorem implies that any risk-averse investor with vN-M preferences will avoid “pure gambles,” in the form of assets with payoffs that simply add more randomness to the payoff of another asset.