ECON 337901
FINANCIAL ECONOMICS

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1 Mathematical and Economic Foundations

- A Mathematical Preliminaries
  - 1 Unconstrained Optimization
  - 2 Constrained Optimization
- B Consumer Optimization
  - 1 Graphical Analysis
  - 2 Algebraic Analysis
  - 3 The Time Dimension
  - 4 The Risk Dimension
- C General Equilibrium
  - 1 Optimal Allocations
  - 2 Equilibrium Allocations

Mathematical Preliminaries

Unconstrained Optimization

$$\max_x F(x)$$

Constrained Optimization

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Unconstrained Optimization

To find the value of x that solves

$$\max_x F(x)$$

you can:

1. Try out every possible value of x.
2. Use calculus.

Since search could take forever, let's use calculus instead.

Unconstrained Optimization

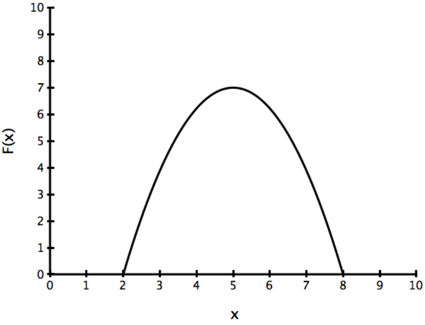
Theorem If x^* solves

$$\max_x F(x),$$

then x^* is a critical point of F, that is,

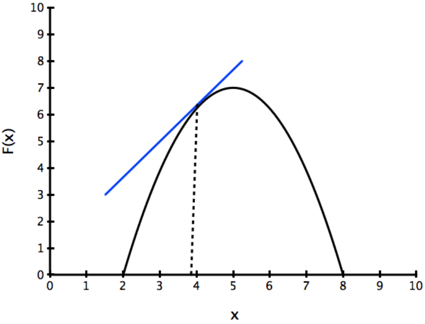
$$F'(x^*) = 0.$$

Unconstrained Optimization



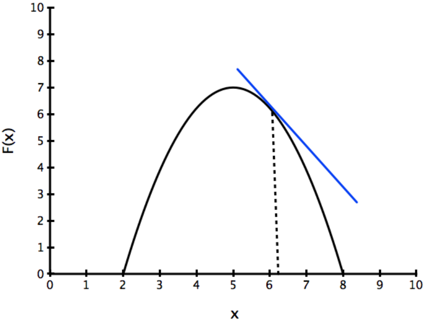
$F(x)$ maximized at $x^* = 5$

Unconstrained Optimization



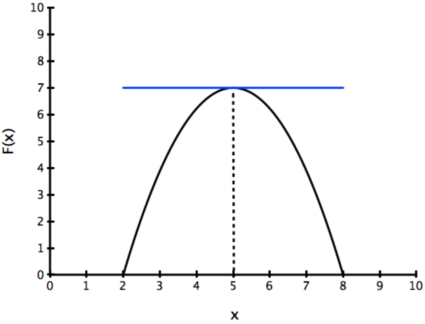
$F'(x) > 0$ when $x < 5$. $F(x)$ can be increased by increasing x.

Unconstrained Optimization



$F'(x) < 0$ when $x > 5$. $F(x)$ can be increased by decreasing x.

Unconstrained Optimization



$F'(x) = 0$ when $x = 5$. $F(x)$ is maximized.

Unconstrained Optimization

Theorem If x^* solves

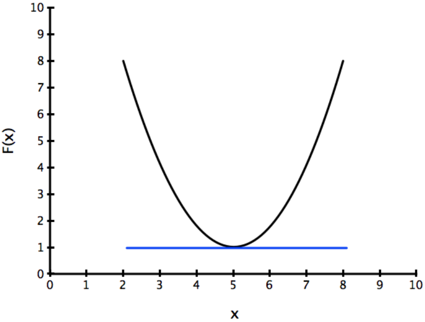
$$\max_x F(x),$$

then x^* is a critical point of F, that is,

$$F'(x^*) = 0.$$

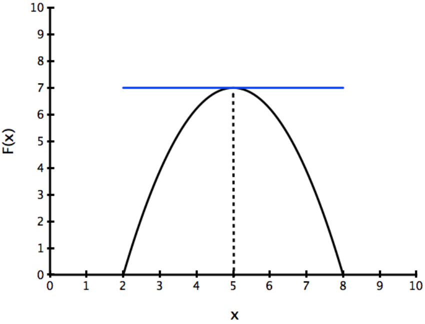
Note that the same first-order necessary condition $F'(x^*) = 0$ also characterizes a value of x^* that minimizes $F(x)$.

Unconstrained Optimization



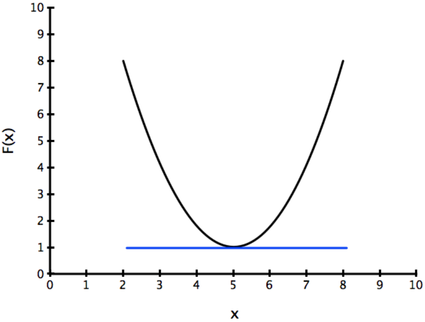
$F'(x) = 0$ when $x = 5$. $F(x)$ is minimized.

Unconstrained Optimization



$F'(x) = 0$ and $F''(x) < 0$ when $x = 5$. $F(x)$ is maximized.

Unconstrained Optimization



$F'(x) = 0$ and $F''(x) > 0$ when $x = 5$. $F(x)$ is minimized.

Unconstrained Optimization

Theorem If x^* solves

$$\max_x F(x),$$

then x^* is a critical point of F, that is,

$$F'(x^*) = 0.$$

Unconstrained Optimization

Theorem If

$$F'(x^*) = 0 \text{ and } F''(x^*) < 0,$$

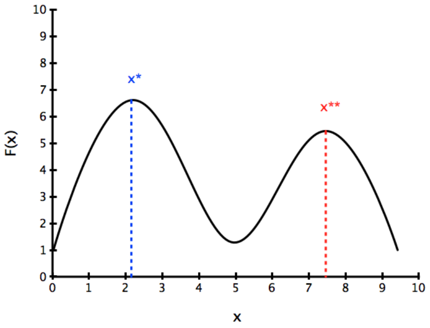
then x^* solves

$$\max_x F(x)$$

(at least locally).

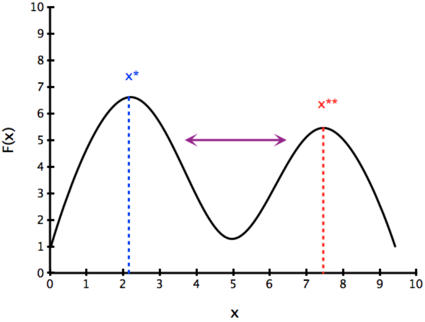
The first-order condition $F'(x^*) = 0$ and the second-order condition $F''(x^*) < 0$ are sufficient conditions for the value of x that (locally) maximizes $F(x)$.

Unconstrained Optimization



$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer x^{**} and
$F'(x^*) = 0$ and $F''(x^*) < 0$ at the global maximizer x^*.

Unconstrained Optimization



$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer x^{**} and $F'(x^*) = 0$ and $F''(x^*) < 0$ at the global maximizer x^*, but $F''(x) > 0$ in between x^* and x^{**}.

Unconstrained Optimization

Theorem If

$$F'(x^*) = 0$$

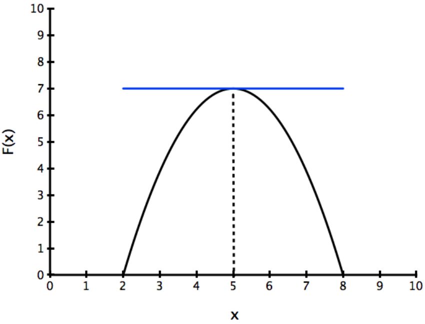
and

$$F''(x) < 0 \text{ for all } x \in \mathbb{R},$$

then x^* solves

$$\max_x F(x).$$

Unconstrained Optimization



$F''(x) < 0$ for all $x \in \mathbb{R}$ and $F'(5) = 0$. $F(x)$ is maximized when $x = 5$.

Unconstrained Optimization

If $F''(x) < 0$ for all $x \in \mathbb{R}$, then the function F is concave.

When F is concave, the first-order condition $F'(x^*) = 0$ is both necessary and sufficient for the value of x that maximizes $F(x)$.

And, as we are about to see, concave functions arise frequently and naturally in economics and finance.

Unconstrained Optimization: Example 1

Consider the problem

$$\max_x \left(-\frac{1}{2}\right)(x-\tau)^2,$$

where τ is a number ($\tau \in \mathbb{R}$) that we might call the "target."

The first-order condition

$$-(x^* - \tau) = 0$$

leads us immediately to the solution: $x^* = \tau$.

Unconstrained Optimization: Example 2

Consider maximizing a function of three variables:

$$\max_{x_1, x_2, x_3} F(x_1, x_2, x_3)$$

Even if each variable can take on only 1,000 values, there are one billion possible combinations of (x_1, x_2, x_3) to search over!

This is an example of what Richard Bellman (US, 1920-1984) called the "curse of dimensionality."

Unconstrained Optimization: Example 2

Consider the problem:

$$\max_{x_1,x_2,x_3} \left(-\frac{1}{2}\right)(x_1-\tau)^2 + \left(-\frac{1}{2}\right)(x_2-x_1)^2 + \left(-\frac{1}{2}\right)(x_3-x_2)^2.$$

Now the three first-order conditions

$$-(x_1^*-\tau)+(x_2^*-x_1^*)=0$$
$$-(x_2^*-x_1^*)+(x_3^*-x_2^*)=0$$
$$-(x_3^*-x_2^*)=0$$

lead us to the solution: $x_1^*=x_2^*=x_3^*=\tau$.

Constrained Optimization

To find the value of x that solves

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

you can:

1. Try out every possible value of x.
2. Use calculus.

Since search could take forever, let's use calculus instead.

Constrained Optimization

A method for solving constrained optimization problems like

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

was developed by Joseph-Louis Lagrange (France/Italy, 1736-1813) and extended by Harold Kuhn (US, 1925-2014) and Albert Tucker (US, 1905-1995).

Constrained Optimization

Associated with the problem:

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Define the Lagrangian

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

where λ is the Lagrange multiplier.

Constrained Optimization

Then, look for a critical point of the full Lagrangian

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

instead of just the objective function F by itself.

That is, use the FOC

$$F'(x^*) - \lambda^* G'(x^*) = 0.$$

Constrained Optimization

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

Theorem (Kuhn-Tucker) If x^* maximizes $F(x)$ subject to $c \geq G(x)$, then there exists a value $\lambda^* \geq 0$ such that, together, x^* and λ^* satisfy the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

and the complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0.$$

Constrained Optimization

In the case where $c > G(x^*)$, the constraint is non-binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

requires that $\lambda^* = 0$.

And the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = 0$.

Constrained Optimization

In the case where $c = G(x^*)$, the constraint is binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

puts no further restriction on $\lambda^* \geq 0$.

Now the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = \lambda^* G'(x^*)$.

Constrained Optimization: Example 1

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 7 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 7$, and $G(x) = x$. The Lagrangian is

$$L(x,\lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x).$$

Constrained Optimization: Example 1

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x),$$

the first-order condition

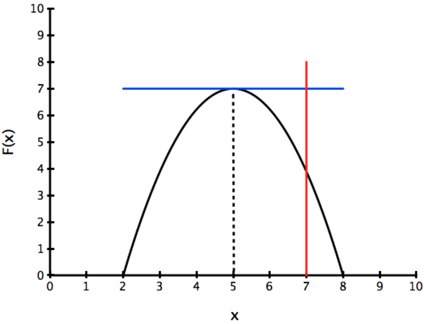
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(7 - x^*) = 0$$

are satisfied with $x^* = 5$, $F'(x^*) = 0$, $\lambda^* = 0$, and $7 > x^*$.

Constrained Optimization: Example 1



Here, the solution has $F'(x^*) = 0$ since the constraint is nonbinding.

Constrained Optimization: Example 2

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 4 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 4$, and $G(x) = x$. The Lagrangian is

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x).$$

Constrained Optimization: Example 2

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x),$$

the first-order condition

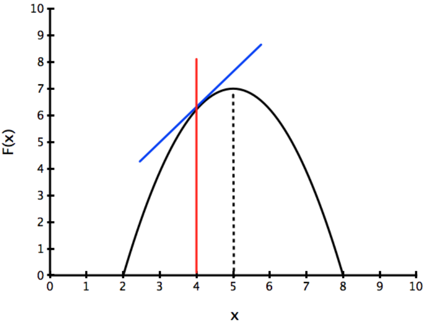
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(4 - x^*) = 0$$

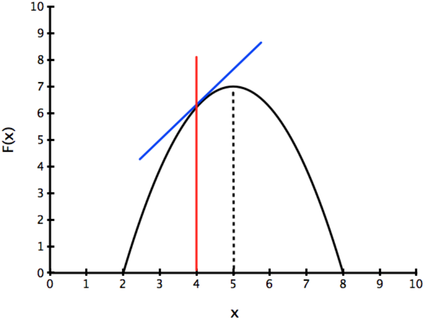
are satisfied with $x^* = 4$ and $F'(x^*) = \lambda^* = 1 > 0$.

Constrained Optimization: Example 2



Here, the solution has $F'(x^*) = \lambda^* G'(x^*) > 0$ since the constraint is binding. $F'(x^*) > 0$ indicates that we'd like to increase the value of x, but the constraint won't let us.

Constrained Optimization: Example 2



With a binding constraint, $F'(x^*) \neq 0$ but $F'(x^*) - \lambda^* G'(x^*) = 0$. The value x^* that solves the problem is a critical point, not of the objective function $F(x)$, but instead of the entire Lagrangian $F(x) + \lambda[c - G(x)]$.

Consumer Optimization

1. Graphical Analysis
2. Algebraic Analysis
3. Time Dimension
4. Risk Dimension

Consumer Optimization

Alfred Marshall, *Principles of Economics*, 1890. – supply and demand

Francis Edgeworth, *Mathematical Psychics*, 1881.

Vilfredo Pareto, *Manual of Political Economy*, 1906. – indifference curves

Consumer Optimization

John Hicks, *Value and Capital*, 1939. – wealth and substitution effects

Paul Samuelson, *Foundations of Economic Analysis*, 1947. – mathematical reformulation

Irving Fisher, *The Theory of Interest*, 1930. – intertemporal extension.

Consumer Optimization

Gerard Debreu, *Theory of Value*, 1959.

Kenneth Arrow, "The Role of Securities in the Optimal Allocation of Risk Bearing," *Review of Economic Studies*, 1964.

Extensions to include risk and uncertainty.

Consumer Optimization: Graphical Analysis

Consider a consumer who likes two goods: apples and bananas.

$$
\begin{aligned}
Y &= \text{income} \\
c_a &= \text{consumption of apples} \\
c_b &= \text{consumption of bananas} \\
p_a &= \text{price of an apple} \\
p_b &= \text{price of a banana}
\end{aligned}
$$

The consumer's budget constraint is

$$Y \geq p_a c_a + p_b c_b$$

Consumer Optimization: Graphical Analysis

So long as the consumer always prefers more to less, the budget constraint will always bind:

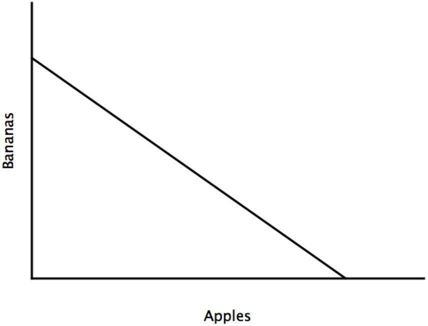
$$Y = p_a c_a + p_b c_b$$

or

$$c_b = \frac{Y}{p_b} - \left(\frac{p_a}{p_b}\right) c_a$$

Which shows that the graph of the budget constraint will be a straight line with slope $-(p_a/p_b)$ and intercept Y/p_b.

Consumer Optimization: Graphical Analysis



The budget constraint is a straight line with slope $-(p_a/p_b)$ and intercept Y/p_b.

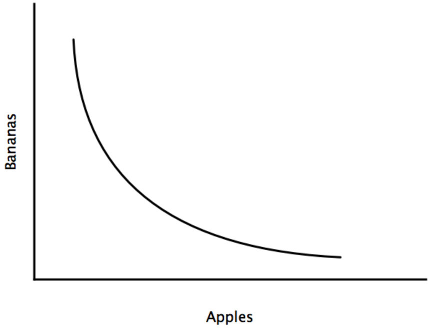
Consumer Optimization: Graphical Analysis

The budget constraint describes the consumer's market opportunities.

Francis Edgeworth (Ireland, 1845-1926) and Vilfredo Pareto (Italy, 1848-1923) were the first to use indifference curves to describe the consumer's preferences.

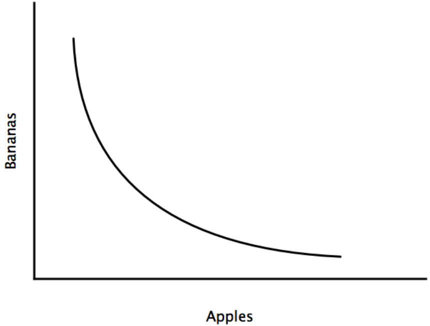
Each indifference curve traces out a set of combinations of apples and bananas that give the consumer a given level of utility or satisfaction.

Consumer Optimization: Graphical Analysis



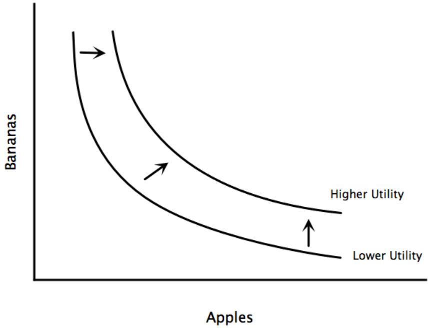
Each indifference curve traces out a set of combinations of apples and bananas that give the consumer a given level of utility.

Consumer Optimization: Graphical Analysis



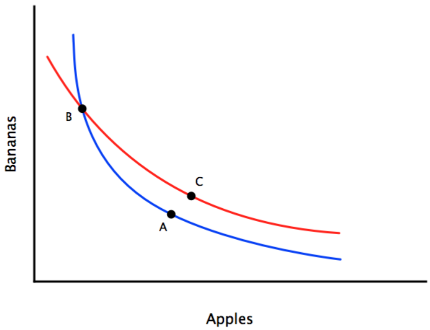
Each indifference curve slopes down, since the consumer requires more apples to compensate for a loss of bananas and more bananas to compensate for a loss of apples, if more is preferred to less.

Consumer Optimization: Graphical Analysis



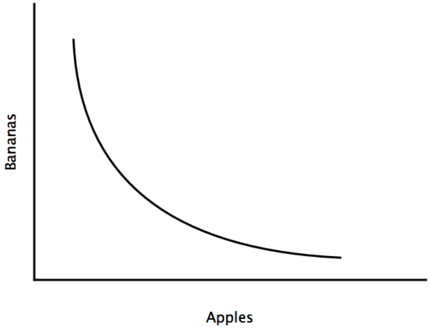
Indifference curves farther away from the origin represent higher levels of utility, if more is preferred to less.

Consumer Optimization: Graphical Analysis



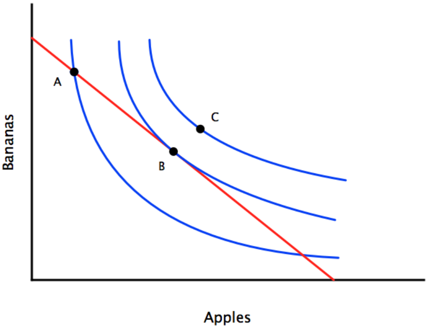
A and B yield the same level of utility, and B and C yield the same level of utility, but C is preferred to A if more is preferred to less. Indifference curves cannot intersect.

Consumer Optimization: Graphical Analysis



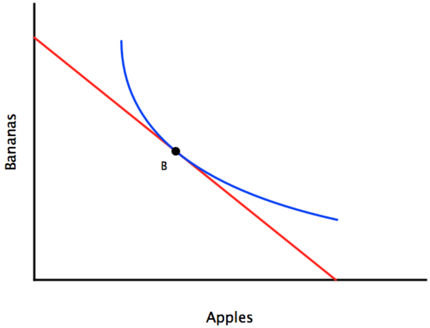
Indifference curves are convex to the origin if consumers have a preference for diversity.

Consumer Optimization: Graphical Analysis



A is suboptimal and C is infeasible. B is optimal.

Consumer Optimization: Graphical Analysis



At B, the optimal choice, the indifference curve is tangent to the budget constraint.

Consumer Optimization: Graphical Analysis

Recall that the budget constraint

$$Y = p_a c_a + p_b c_b$$

or

$$c_b = \frac{Y}{p_b} - \left(\frac{p_a}{p_b}\right) c_a$$

has slope $-(p_a/p_b)$.

Consumer Optimization: Graphical Analysis

Suppose that the consumer's preferences are also described by the utility function

$$u(c_a) + \beta u(c_b).$$

The function u is increasing, with $u'(c) > 0$, so that more is preferred to less, and concave, with $u''(c) < 0$, so that marginal utility falls as consumption rises.

The parameter β measures how much more (if $\beta > 1$) or less (if $\beta < 1$) the consumer likes bananas compared to apples.

Consumer Optimization: Graphical Analysis

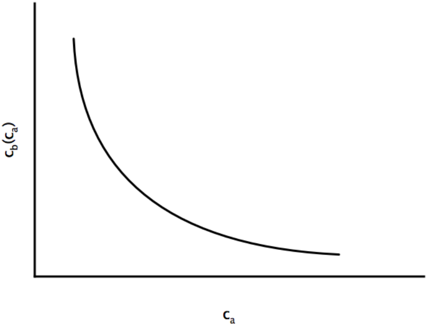
Since an indifference curve traces out the set of (c_a, c_b) combinations that yield a given level of utility $\bar{U}$, the equation for an indifference curve is

$$\bar{U} = u(c_a) + \beta u(c_b).$$

Use this equation to define a new function, $c_b(c_a)$, describing the number of bananas needed, for each number of apples, to keep the consumer on this indifference curve:

$$\bar{U} = u(c_a) + \beta u[c_b(c_a)].$$

Consumer Optimization: Graphical Analysis



The function $c_b(c_a)$ satisfies $\bar{U} = u(c_a) + \beta u[c_b(c_a)]$.

Consumer Optimization: Graphical Analysis

Differentiate both sides of

$$\bar{U} = u(c_a) + \beta u[c_b(c_a)]$$

to obtain

$$0 = u'(c_a) + \beta u'[c_b(c_a)]c_b'(c_a)$$

or

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]}.$$

Consumer Optimization: Graphical Analysis

This last equation,

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]},$$

written more simply as

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'(c_b)},$$

measures the slope of the indifference curve: the consumer's marginal rate of substitution.

Consumer Optimization: Graphical Analysis

Thus, the tangency of the budget constraint and indifference curve can be expressed mathematically as

$$\frac{p_a}{p_b} = \frac{u'(c_a)}{\beta u'(c_b)}.$$

The marginal rate of substitution equals the relative prices.

Consumer Optimization: Graphical Analysis

Returning to the more general expression

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]},$$

we can see that $c_b'(c_a) < 0$, so that the indifference curve is downward-sloping, so long as the utility function u is strictly increasing, that is, if more is preferred to less.

Consumer Optimization: Graphical Analysis

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]}$$

Differentiating again yields

$$c_b''(c_a) = -\frac{\beta u'[c_b(c_a)]u''(c_a) - u'(c_a)\beta u''[c_b(c_a)]c_b'(c_a)}{\{\beta u'[c_b(c_a)]\}^2},$$

which is positive if u is strictly increasing (more is preferred to less) and concave (diminishing marginal utility). In this case, the indifference curve will be convex. Again, we see how concave functions have mathematical properties and economic implications that we like.

Consumer Optimization: Algebraic Analysis

Graphical analysis works fine with two goods.

But what about three goods? That depends on how good an artist you are!

And what about four or more goods? Our universe won't accommodate a graph like that!

But once again, calculus makes it easier!

Consumer Optimization: Algebraic Analysis

Consider a consumer who likes three goods:

$$Y = \text{income}$$
$$c_i = \text{consumption of goods } i = 0, 1, 2$$
$$p_i = \text{price of goods } i = 0, 1, 2$$

Suppose the consumer's utility function is

$$u(c_0) + \alpha u(c_1) + \beta u(c_2),$$

where α and β are weights on goods 1 and 2 relative to good 0.

Consumer Optimization: Algebraic Analysis

The consumer chooses c_0, c_1, and c_2 to maximize the utility function

$$u(c_0) + \alpha u(c_1) + \beta u(c_2),$$

subject to the budget constraint

$$Y \geq p_0 c_0 + p_1 c_1 + p_2 c_2.$$

The Lagrangian for this problem is

$$L = u(c_0) + \alpha u(c_1) + \beta u(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$

Consumer Optimization: Algebraic Analysis

$$L = u(c_0) + \alpha u(c_1) + \beta u(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$

First-order conditions:

$$u'(c_0^*) - \lambda^* p_0 = 0$$

$$\alpha u'(c_1^*) - \lambda^* p_1 = 0$$

$$\beta u'(c_2^*) - \lambda^* p_2 = 0$$

Consumer Optimization: Algebraic Analysis

The first-order conditions

$$u'(c_0^*) - \lambda^* p_0 = 0$$

$$\alpha u'(c_1^*) - \lambda^* p_1 = 0$$

$$\beta u'(c_2^*) - \lambda^* p_2 = 0$$

imply

$$\frac{u'(c_0^*)}{\alpha u'(c_1^*)} = \frac{p_0}{p_1} \text{ and } \frac{u'(c_0^*)}{\beta u'(c_2^*)} = \frac{p_0}{p_2} \text{ and } \frac{\alpha u'(c_1^*)}{\beta u'(c_2^*)} = \frac{p_1}{p_2}.$$

The marginal rate of substitution equals the relative prices.

Consumer Optimization: The Time Dimension

Irving Fisher (US, 1867-1947) was the first to recognize that the basic theory of consumer decision-making could be used to understand how to optimally allocate spending intertemporally, that is, over time, as well as how to optimally allocate spending across different goods in a static, or point-in-time, analysis.

Consumer Optimization: The Time Dimension

Following Fisher, return to the case of two goods, but reinterpret:

$$c_0 = \text{consumption today}$$

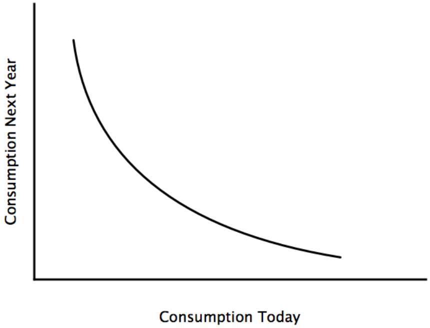
$$c_1 = \text{consumption next year}$$

Suppose that the consumer's utility function is

$$u(c_0) + \beta u(c_1),$$

where β now has a more specific interpretation, as the discount factor, a measure of patience.

Consumer Optimization: The Time Dimension



A concave utility function implies that indifference curves are convex, so that the consumer has a preference for a smoothness in consumption.

Consumer Optimization: The Time Dimension

Next, let

$$Y_0 = \text{income today}$$
$$Y_1 = \text{income next year}$$
$$s = \text{amount saved (or borrowed if negative) today}$$
$$r = \text{interest rate}$$

Consumer Optimization: The Time Dimension

Today, the consumer divides his or her income up into an amount to be consumed and an amount to be saved:

$$Y_0 \geq c_0 + s.$$

Next year, the consumer simply spends his or her income, including interest earnings if s is positive or net of interest expenses if s is negative:

$$Y_1 + (1 + r)s \geq c_1.$$

Consumer Optimization: The Time Dimension

Divide both sides of next year's budget constraint by $1 + r$ to get

$$\frac{Y_1}{1+r} + s \geq \frac{c_1}{1+r}.$$

Now combine this inequality with this year's budget constraint

$$Y_0 \geq c_0 + s.$$

to get

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

Consumer Optimization: The Time Dimension

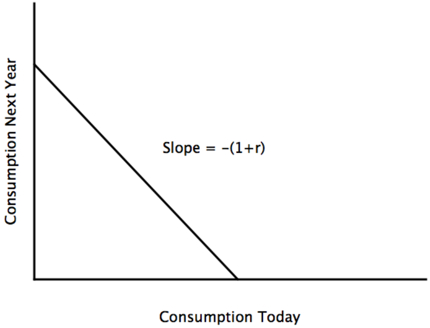
The "lifetime" budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}$$

says that the present value of income must be sufficient to cover the present value of consumption over the two periods. It also shows that the "price" of consumption today relative to the "price" of consumption next year is related to the interest rate via

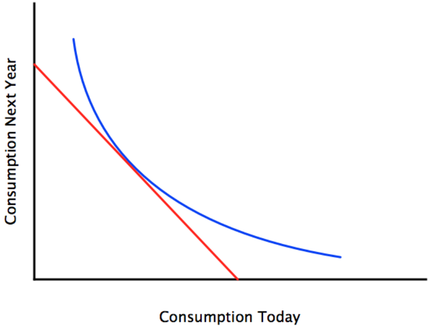
$$\frac{p_0}{p_1} = 1 + r.$$

Consumer Optimization: The Time Dimension



The slope of the intertemporal budget constraint is $-(1 + r)$.

Consumer Optimization: The Time Dimension



At the optimum, the intertemporal marginal rate of substitution equals the slope of the intertemporal budget constraint.

Consumer Optimization: The Time Dimension

We now know the answer ahead of time: if we take an algebraic approach to solve the consumer's problem, we will find that the IMRS equals the slope of the intertemporal budget constraint:

$$\frac{u'(c_0)}{\beta u'(c_1)} = 1 + r.$$

But let's use calculus to derive the same result.

Consumer Optimization: The Time Dimension

The problem is to choose c_0 and c_1 to maximize utility

$$u(c_0) + \beta u(c_1)$$

subject to the budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

The Lagrangian is

$$L = u(c_0) + \beta u(c_1) + \lambda \left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

Consumer Optimization: The Time Dimension

$$L = u(c_0) + \beta u(c_1) + \lambda \left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

The first-order conditions

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta u'(c_1^*) - \lambda^* \left(\frac{1}{1+r} \right) = 0.$$

lead directly to the graphical result

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r.$$

Consumer Optimization: The Time Dimension

At first glance, Fisher's model seems unrealistic, especially in its assumption that the consumer can borrow at the same interest rate r that he or she receives on his or her savings.

A reinterpretation of saving and borrowing in this framework, however, can make it more applicable, at least for some consumers.

Investment Strategies and Cash Flows

Investment Strategy	Cash Flow at $t = 0$	Cash Flow at $t = 1$
Saving	-1	$+(1+r)$
Buying a bond (long position in bonds)	-1	$+(1+r)$

Investment Strategies and Cash Flows

Investment Strategy	Cash Flow at $t = 0$	Cash Flow at $t = 1$
Borrowing	$+1$	$-(1 + r)$
Issuing a bond	$+1$	$-(1 + r)$
Short selling a bond (short position in bonds)	$+1$	$-(1 + r)$
Selling a bond (out of inventory)	$+1$	$-(1 + r)$

Investment Strategies and Cash Flows

Investment Strategy	Cash Flow at $t = 0$	Cash Flow at $t = 1$
Buying a stock (long position in stocks)	$-P_0^s$	$+P_1^s$
Short selling a stock (short position in stocks)	$+P_0^s$	$-P_1^s$
Selling a stock (out of inventory)	$+P_0^s$	$-P_1^s$

Consumer Optimization: The Time Dimension

Someone who already owns bonds can “borrow” by selling a bond out of inventory. In fact, theories like Fisher’s work better when applied to consumers who already own stocks and bonds.

Greg Mankiw and Stephen Zeldes, “The Consumption of Stockholders and Nonstockholders,” *Journal of Finance*, 1991.

Annette Vissing-Jorgensen, “Limited Asset Market Participation and the Elasticity of Intertemporal Substitution,” *Journal of Political Economy*, 2002.

Consumer Optimization: The Risk Dimension

In the 1950s and 1960s, Kenneth Arrow (US, 1921-2017, Nobel Prize 1972) and Gerard Debreu (France, 1921-2004, Nobel Prize 1983) extended consumer theory to accommodate risk and uncertainty.

To do so, they drew on earlier ideas developed by others, but added important insights of their own.

Building Blocks of Arrow-Debreu Theory

1. Fisher's (1930) intertemporal model of consumer decision-making.
2. From probability theory: uncertainty described with reference to "states of the world." (Andrey Kolmogorov, 1930s).
3. Expected utility theory (John von Neumann and Oskar Morgenstern, 1947).
4. Contingent claims – stylized financial assets – a powerful analytic device of their own invention.

Consumer Optimization: The Risk Dimension

To be more specific about the source of risk, let's suppose that there are two possible outcomes for income next year, good and bad:

$$Y_0 = \text{income today}$$

$$Y_1^G = \text{income next year in the ``good'' state}$$

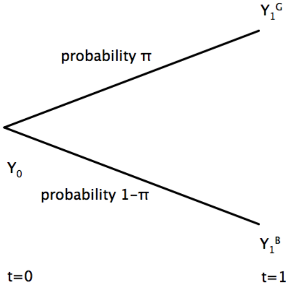
$$Y_1^B = \text{income next year in the ``bad'' state}$$

where the assumption $Y_1^G > Y_1^B$ makes the "good" state good and where

$$\pi = \text{probability of the good state}$$

$$1 - \pi = \text{probability of the bad state}$$

Consumer Optimization: The Risk Dimension



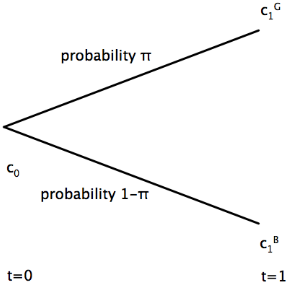
An event tree highlights randomness in income as the source of risk.

Consumer Optimization: The Risk Dimension

Arrow and Debreu used the probabilistic idea of states of the world to extend Irving Fisher's work, recognizing that under these circumstances, the consumer chooses between three goods:

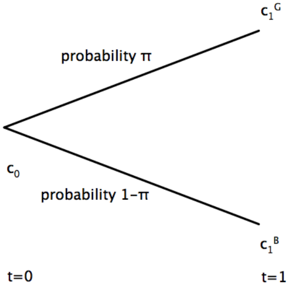
$$c_0 = \text{consumption today}$$
$$c_1^G = \text{consumption next year in the good state}$$
$$c_1^B = \text{consumption next year in the bad state}$$

Consumer Optimization: The Risk Dimension



Under uncertainty, the consumer chooses consumption today and consumption in both states next year.

Consumer Optimization: The Risk Dimension



Uncertainty about future income "induces" randomness in future consumption as well.

Consumer Optimization: The Risk Dimension

Suppose that the consumer's utility function is

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

so that the terms involving next year's consumption are weighted by the probability that each state will occur as well as by the discount factor β.

Consumer Optimization: The Risk Dimension

In probability theory, if a random variable X can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of X is

$$E(X) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n.$$

Consumer Optimization: The Risk Dimension

Hence, by assuming that the consumer's utility function is

$$u(c_0) + \beta \pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

we are assuming that the consumer's seeks to maximize expected utility

$$u(c_0) + \beta E[u(c_1)].$$

Consumer Optimization: The Risk Dimension

But by writing out all three terms,

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

we can see that concavity of the function u, which in the standard microeconomic case represents a preference for diversity, represents here a preference for smoothness in consumption over time and across states in the future – the consumer is risk averse in the sense that he or she does not want consumption in the bad state to be too much different from consumption in the good state.

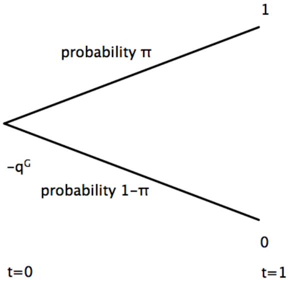
Consumer Optimization: The Risk Dimension

To implement these state-contingent consumption plans, Arrow and Debreu imagined that the consumer would trade contingent claims for both future states.

A contingent claim for the good state costs q^G today, and delivers one unit of consumption next year in the good state and zero units of consumption next year in the bad state.

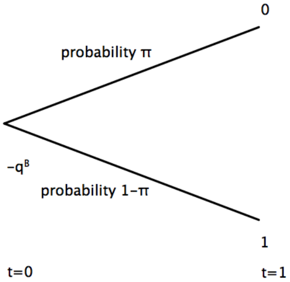
A contingent claim for the bad state costs q^B today, and delivers one unit of consumption next year in the bad state and zero units of consumption next year in the good state.

Consumer Optimization: The Risk Dimension



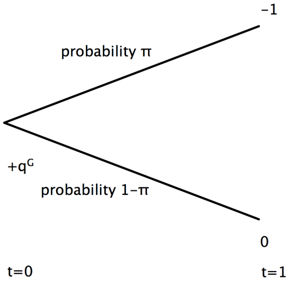
Payoffs for the contingent claim for the good state (a long position).

Consumer Optimization: The Risk Dimension



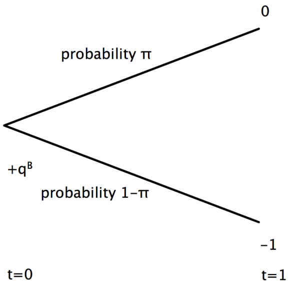
Payoffs for the contingent claim for the bad state (a long position).

Consumer Optimization: The Risk Dimension



Payoffs for a short position in the contingent claim for the good state.

Consumer Optimization: The Risk Dimension



Payoffs for a short position in the contingent claim for the bad state.

Consumer Optimization: The Risk Dimension

Trading Strategy	Claim	Cash Flow at $t = 0$	Cash Flow in Good State at $t = 1$	Cash Flow in Bad State at $t = 1$
Long	Good	$-q^G$	$+1$	0
Long	Bad	$-q^B$	0	$+1$
Short	Good	$+q^G$	-1	0
Short	Bad	$+q^B$	0	-1

Like a sophisticated form of saving and borrowing, where the investor can "fine-tune" the future state in which payments are received or made.

Consumer Optimization: The Risk Dimension

Today, the consumer divides his or her income up into an amount to be consumed and amounts used to purchase the two contingent claims:

$$Y_0 \geq c_0 + q^G s^G + q^B s^B,$$

where s^G and s^B denote the number of each contingent claim purchased or sold short.

If either s^G or s^B is negative, the consumer is taking a short position in that claim.

Consumer Optimization: The Risk Dimension

Next year, the consumer simply spends his or her income, including payoffs on contingent claims:

$$Y_1^G + s^G \geq c_1^G$$

in the good state and

$$Y_1^B + s^B \geq c_1^B$$

in the bad state.

Consumer Optimization: The Risk Dimension

$$Y_0 \geq c_0 + q^G s^G + q^B s^B$$
$$Y_1^G + s^G \geq c_1^G$$
$$Y_1^B + s^B \geq c_1^B$$

Multiply both sides of the second equation by q^G and both sides of the third equation by q^B, Then add them all up to get the lifetime budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

Consumer Optimization: The Risk Dimension

The problem is to choose c_0, c_1^G, and c_1^B to maximize expected utility

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

subject to the budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

This was Arrow and Debreu's key insight: that finance is like grocery shopping. Mathematically, making decisions over time and under uncertainty is no different from choosing apples, bananas, and pears!

Consumer Optimization: The Risk Dimension

The Lagrangian is

$$
\begin{aligned}
L &= u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B) \\
&\quad + \lambda \left(Y_0 + q^G Y_1^G + q^B Y_1^B - c_0 - q^G c_1^G - q^B c_1^B \right),
\end{aligned}
$$

and the first-order conditions are

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi) u'(c_1^{B*}) - \lambda^* q^B = 0$$

Consumer Optimization: The Risk Dimension

The first-order conditions

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi)u'(c_1^{B*}) - \lambda^* q^B = 0$$

imply that marginal rates of substitution equal relative prices:

$$\frac{u'(c_0^*)}{\beta\pi u'(c_1^{G*})} = \frac{1}{q^G} \text{ and } \frac{u'(c_0^*)}{\beta(1-\pi)u'(c_1^{B*})} = \frac{1}{q^B}$$

$$\text{and } \frac{\pi u'(c_1^{G*})}{(1-\pi)u'(c_1^{B*})} = \frac{q^G}{q^B}.$$

Consumer Optimization: The Risk Dimension

Do we really observe consumers trading in contingent claims?

Yes, if we think of financial assets as "bundles" of contingent claims.

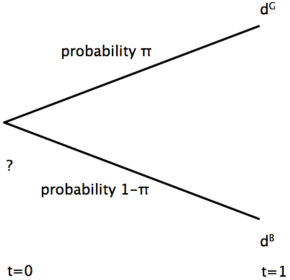
This insight is also Arrow and Debreu's.

Consumer Optimization: The Risk Dimension

A "stock" is a risky asset that pays dividend d^G next year in the good state and d^B next year in the bad state.

These payoffs can be replicated by buying d^G contingent claims for the good state and d^B contingent claims for the bad state.

Consumer Optimization: The Risk Dimension



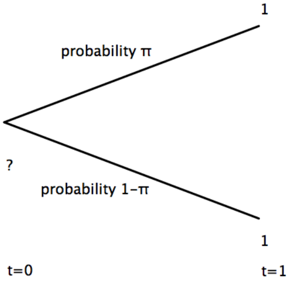
Payoffs for the stock.

Consumer Optimization: The Risk Dimension

A "bond" is a safe asset that pays off one next year in the good state and one next year in the bad state.

These payoffs can be replicated by buying one contingent claim for the good state and one contingent claim for the bad state.

Consumer Optimization: The Risk Dimension



Payoffs for the bond.

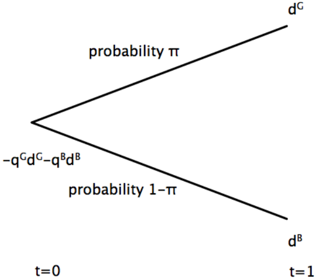
Consumer Optimization: The Risk Dimension

If we start with knowledge of the contingent claims prices q^G and q^B, then we can infer that the stock must sell today for

$$q^{stock} = q^G d^G + q^B d^B.$$

Since if the stock cost more than the equivalent bundle of contingent claims, traders could make profits for sure by short selling the stock and buying the contingent claims; and if the stock cost less than the equivalent bundle of contingent claims, traders could make profits for sure by buying the stock and selling the contingent claims.

Consumer Optimization: The Risk Dimension



"Pricing" the stock.

Consumer Optimization: The Risk Dimension

Likewise, if we start with knowledge of the contingent claims prices q^G and q^B, then we can infer that the bond must sell today for

$$q^{bond} = q^G + q^B.$$

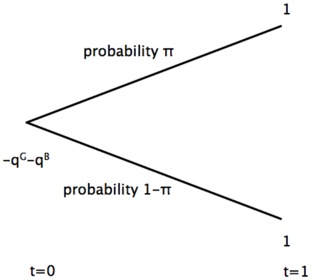
Since the bond pays off one for sure next year, the interest rate, defined as the return on the risk-free bond, is

$$1 + r = \frac{1}{q^{bond}} = \frac{1}{q^G + q^B}.$$

The bond price relates to the interest rate via

$$q^{bond} = \frac{1}{1 + r}.$$

Consumer Optimization: The Risk Dimension



Pricing the bond.

Consumer Optimization: The Risk Dimension

We've already seen how contingent claims can be used to replicate the stock and the bond.

Now let's see how the stock and the bond can be used to replicate the contingent claims.

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the good state.

In the good state, the payoffs should be

$$sd^G + b = 1$$

and in the bad state, the payoffs should be

$$sd^B + b = 0$$

since the contingent claim pays off one in the good state and zero in the bad state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the good state:

$$sd^G + b = 1$$

$$sd^B + b = 0 \Rightarrow b = -sd^B$$

Substitute the second equation into the first to solve for

$$s = \frac{1}{d^G - d^B} \text{ and } b = \frac{-d^B}{d^G - d^B}$$

Since s and b are of opposite sign, this requires going "long" one asset and "short" the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the good state:

$$s = \frac{1}{d^G - d^B} \text{ and } b = \frac{-d^B}{d^G - d^B}$$

If we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the good state would have price

$$q^G = q^{stock} s + q^{bond} b = \frac{q^{stock} - d^B q^{bond}}{d^G - d^B}.$$

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the bad state.

In the good state, the payoffs should be

$$sd^G + b = 0$$

and in the bad state, the payoffs should be

$$sd^B + b = 1$$

since the contingent claim pays off one in the bad state and zero in the good state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$sd^G + b = 0 \Rightarrow b = -sd^G$$

$$sd^B + b = 1$$

Substitute the first equation into the second to solve for

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, this requires going long one asset and short the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, if we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the bad state would have price

$$q^B = q^{stock} s + q^{bond} b = \frac{d^G q^{bond} - q^{stock}}{d^G - d^B}.$$

Consumer Optimization: The Risk Dimension

What makes it possible to go back and forth between traded assets, like stocks and bonds, and contingent claims is that there are the same number of traded assets as there are possible states of the world next year.

More generally, asset markets are complete if there are as many assets (with linearly independent payoffs) as there are states next year.

Consumer Optimization: The Risk Dimension

If asset markets are complete, then we can use the prices of traded assets to infer the prices of contingent claims.

Then we can use the contingent claims prices to infer the price of any newly-introduced asset.

General Equilibrium

An allocation of resources is Pareto optimal if it is impossible to reallocate those resources without making at least one consumer worse off.

A competitive equilibrium is an allocation of resources and a set of prices such that, at those prices: (i) each consumer is maximizing utility subject to his or her budget constraint and (ii) the supply of each good equal the demand for each good.

The two welfare theorems of economics link optimal and equilibrium allocations.

Optimal Allocations

In an economy with two consumers, 1 and 2, and two goods, a and b, the key properties of Pareto optimal allocations can be illustrated using an Edgeworth box.

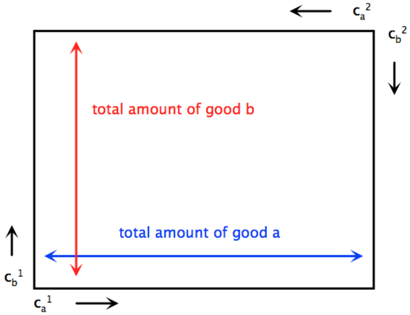
$c_a^1 =$ 1's consumption of good a

$c_b^1 =$ 1's consumption of good b

$c_a^2 =$ 2's consumption of good a

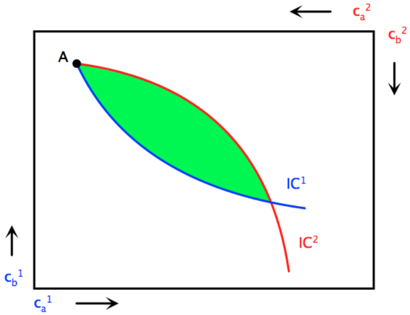
$c_b^2 =$ 2's consumption of good b

Optimal Allocations



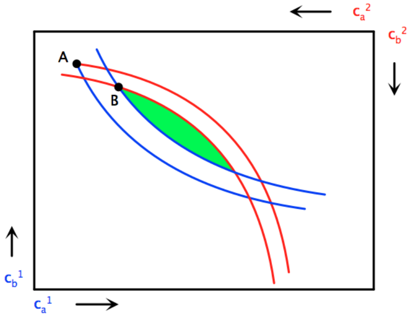
The Edgeworth box contains the entire set of feasible allocations.

Optimal Allocations



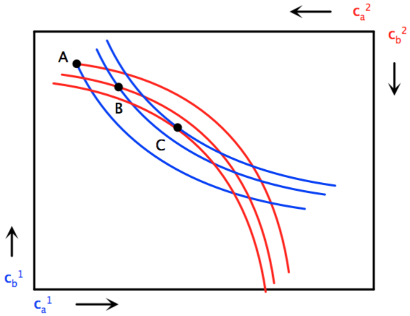
Both consumers prefer allocations in the green region to A.

Optimal Allocations



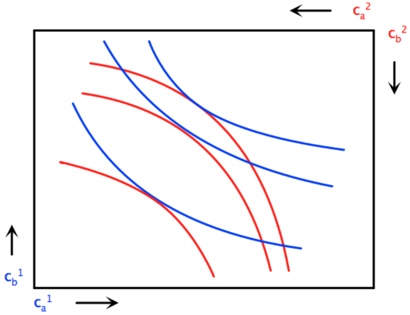
Both consumers prefer B to A, but still there are allocations that are even more strongly preferred.

Optimal Allocations



At C, there is no way to make one consumer better off without making the other worse off. C is Pareto optimal.

Optimal Allocations



There are many Pareto optimal allocations, but each is characterized by the tangency of the two consumers' indifference curves.

Optimal Allocations

Note that Pareto optimality is a welfare criterion that accounts for efficiency but not equity: an allocation may be Pareto optimal even though it provides most of the goods to one consumer.

But since the slope of the indifference curves is measured by the marginal rate of substitution, the mathematical condition associated with all Pareto optimal allocations is

$$MRS^1_{a,b} = MRS^2_{a,b}. \tag{PO}$$

Optimal Allocations

Suppose that consumer 1 has utility function

$$u(c_a^1) + \alpha u(c_b^1)$$

and consumer 2 has utility function

$$v(c_a^2) + \beta v(c_b^2).$$

Optimal Allocations

Consider a benevolent "social planner," who divides Y_a units of good a and Y_b units of good b up between the two consumers, subject to the resource constraints

$$Y_a \geq c_a^1 + c_a^2$$

and

$$Y_b \geq c_b^1 + c_b^2,$$

so as to maximize a weighted average of their utilities:

$$\theta[u(c_a^1) + \alpha u(c_b^1)] + (1-\theta)[v(c_a^2) + \beta v(c_b^2)],$$

where $1 > \theta > 0$.

Optimal Allocations

Since there are two constraints, the Lagrangian for the social planner's problem requires two Lagrange multipliers:

$$
\begin{aligned}
L &= \theta[u(c_a^1) + \alpha u(c_b^1)] + (1-\theta)[v(c_a^2) + \beta v(c_b^2)] \\
&\quad + \lambda_a(Y_a - c_a^1 - c_a^2) + \lambda_b(Y_b - c_b^1 - c_b^2).
\end{aligned}
$$

The first-order conditions are:

$$\theta u'(c_a^1) - \lambda_a = 0$$

$$\theta \alpha u'(c_b^1) - \lambda_b = 0$$

$$(1-\theta) v'(c_a^2) - \lambda_a = 0$$

$$(1-\theta)\beta v'(c_b^2) - \lambda_b = 0.$$

Optimal Allocations

The first-order conditions

$$\theta u'(c_a^1) - \lambda_a = 0$$

$$\theta \alpha u'(c_b^1) - \lambda_b = 0$$

$$(1-\theta) v'(c_a^2) - \lambda_a = 0$$

$$(1-\theta) \beta v'(c_b^2) - \lambda_b = 0.$$

imply that

$$\frac{u'(c_a^1)}{\alpha u'(c_b^1)} = \frac{\lambda_a}{\lambda_b} = \frac{v'(c_a^2)}{\beta v'(c_b^2)},$$

a restatement of (PO) that must hold for any value of θ.

Equilibrium Allocations

Now let's see what happens when markets, instead of a social planner, allocate resources:

$$Y_a^1 = \text{consumer 1's endowment of good a}$$
$$Y_b^1 = \text{consumer 1's endowment of good b}$$
$$Y_a^2 = \text{consumer 2's endowment of good a}$$
$$Y_b^2 = \text{consumer 2's endowment of good b}$$
$$p_a = \text{price of good a}$$
$$p_b = \text{price of good b}$$

Equilibrium Allocations

Consumer 1 chooses c_a^1 and c_b^1 to maximize utility

$$u(c_a^1) + \alpha u(c_b^1)$$

subject to the budget constraint

$$p_a Y_a^1 + p_b Y_b^1 \geq p_a c_a^1 + p_b c_b^1,$$

taking the prices p_a and p_b as given.

Equilibrium Allocations

The Lagrangian for consumer 1's problem is

$$L = u(c_a^1) + \alpha u(c_b^1) + \lambda^1 (p_a Y_a^1 + p_b Y_b^1 - p_a c_a^1 - p_b c_b^1).$$

The first-order conditions

$$u'(c_a^1) - \lambda^1 p_a = 0$$

$$\alpha u'(c_b^1) - \lambda^1 p_b = 0$$

imply that

$$\frac{u'(c_a^1)}{\alpha u'(c_b^1)} = \frac{p_a}{p_b}. \tag{CE-1}$$

Equilibrium Allocations

Similarly, consumer 2 chooses c_a^2 and c_b^2 to maximize utility

$$v(c_a^2) + \beta v(c_b^2)$$

subject to the budget constraint

$$p_a Y_a^2 + p_b Y_b^2 \geq p_a c_a^2 + p_b c_b^2,$$

taking the prices p_a and p_b as given.

Equilibrium Allocations

The Lagrangian for consumer 2's problem is

$$L = v(c_a^2) + \beta v(c_b^2) + \lambda^2(p_a Y_a^2 + p_b Y_b^2 - p_a c_a^2 - p_b c_b^2).$$

The first-order conditions

$$v'(c_a^2) - \lambda^2 p_a = 0$$

$$\beta v'(c_b^2) - \lambda^2 p_b = 0$$

imply that

$$\frac{v'(c_a^2)}{\beta v'(c_b^2)} = \frac{p_a}{p_b}. \qquad \text{(CE-2)}$$

Equilibrium Allocations

Hence, in any competitive equilibrium

$$\frac{u'(c_a^1)}{\alpha u'(c_b^1)} = \frac{p_a}{p_b}. \tag{CE-1}$$

and

$$\frac{v'(c_a^2)}{\beta v'(c_b^2)} = \frac{p_a}{p_b} \tag{CE-2}$$

must hold, so that

$$\frac{u'(c_a^1)}{\alpha u'(c_b^1)} = \frac{p_a}{p_b} = \frac{v'(c_a^2)}{\beta v'(c_b^2)}.$$

Equilibrium Allocations

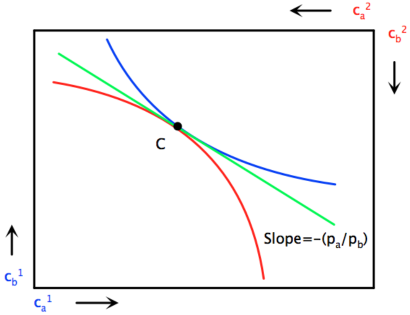
There will be different equilibrium allocations associated with different patterns for the endowments Y_a^1, Y_b^1, Y_a^2, and Y_b^2.

In addition, different equilibrium allocations may require different prices p_a and p_b to equate the supply and demand of each good.

But all equilibrium allocations must satisfy

$$MRS^1_{a,b} = \frac{p_a}{p_b} = MRS^2_{a,b}. \quad \text{(CE)}$$

Equilibrium Allocations



The Pareto optimal allocation C is supported in a competitive equilibrium with prices p_a and p_b, and the equilibrium allocation C is Pareto optimal.

General Equilibrium

The coincidence between (PO) and (CE) underlies results that extend Adam Smith's (Scotland, 1723-1790) notion of an "invisible hand" that guides self-interested individuals to choose resource allocations that are Pareto optimal.

First Welfare Theorem of Economics The resource allocation from a competitive equilibrium is Pareto optimal.

Second Welfare Theorem of Economics A Pareto optimal resource allocation can be supported in a competitive equilibrium.