1. Stocks, Bonds, and Contingent Claims

Consider an economic environment with risk in which there are two periods, \( t = 0 \) and \( t = 1 \), and two possible states at \( t = 1 \): a “good” state that occurs with probability \( \pi = 1/2 \) and a “bad” state that occurs with probability \( 1 - \pi = 1/2 \). Suppose that investors trade two assets in this economy. The first is a “stock,” which sells for \( q_s = 1 \) at \( t = 0 \) and pays a large dividend of \( d^G = 3 \) in the good state at \( t = 1 \) and a small dividend of \( d^B = 1 \) in the bad state at \( t = 1 \). The second is a “bond,” which sells for \( q_b = 0.8 \) at \( t = 0 \) and makes a payoff of one for sure, in both states, at \( t = 1 \).

a. Assume that investors can take long and short positions in both assets. Given the payoffs for the stock and bond, find the combination of stock and bond holdings \( s \) and \( b \) that replicate the payoff provided by a contingent claim for the good state: one in good state and zero in the bad.

b. Next, find the combination of stock and bond holdings that replicate the payoff provided by the contingent claim for the bad state: one in the bad state and zero in the good.

c. Given the answers you derived for parts (a) and (b), above, find the prices at which each contingent claim should sell at \( t = 0 \).
2. Concepts of Dominance

Consider two assets, with percentage returns \( \tilde{R}_1 \) and \( \tilde{R}_2 \) that vary across three states that occur with equal probability as follows:

\[
\begin{array}{c|c|c|c}
\text{State} & \text{Return on} & \pi_1 = 1/3 & \pi_1 = 1/3 & \pi_1 = 1/3 \\
\hline
\text{State 1} & \tilde{R}_1 & 10 & 0 & 10 \\
\text{State 2} & \tilde{R}_2 & 0 & 10 & 20 \\
\end{array}
\]

a. Does one asset exhibit state-by-state dominance over the other? If so, which one?
b. Does one asset exhibit mean-variance dominance over the other? If so, which one?
c. Which asset has the highest Sharpe ratio?

3. Expected Utility and Risk Aversion

Consider an investor with initial income equal to \( Y = 10 \), who must choose between two assets. The first is a risky asset, let’s call it a “stock,” that pays off \( d^G = 3 \) in a good state that occurs with probability \( \pi = 1/2 \) and pays off \( d^B = 0 \) (nothing) in a bad state that occurs with probability \( 1 - \pi = 1/2 \). The second is a safe asset, let’s call it a “bond,” that pays off \( r = 1 \) no matter what, in both the good and bad states of the world. Thus, if the investor chooses the stock, his or her total income will be \( Y + d^G = 13 \) with probability \( 1/2 \) and \( Y + d^B = 10 \) with probability \( 1/2 \), whereas if the investor chooses the bond, his or her total income will be \( Y + r = 11 \) for sure. Assume that this investor has a von Neumann-Morgenstern expected utility function, with a Bernoulli utility function of the constant relative risk aversion (CRRA) form

\[
u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}.
\]

a. Suppose first that the investor’s constant coefficient of relative risk aversion is \( \gamma = 1/2 \). Which asset will he or she prefer: the stock or the bond?
b. Suppose instead that the investor’s constant coefficient of relative risk aversion is \( \gamma = 2 \). Which asset will he or she prefer: the stock or the bond?
c. Go back to assuming that \( \gamma = 1/2 \), but suppose that the investor’s Bernoulli utility function is

\[
v(c) = \frac{3(c^{1-\gamma} - 1)}{1 - \gamma},
\]

instead of the simpler function \( u(c) \) shown before, Which asset will the investor prefer now: the stock or the bond?
4. Certainty Equivalents and Risk Premia

Consider the same investor described in problem 3, above, who has initial income equal to \( Y = 10 \), but focus more specifically on the stock, which pays off \( d^G = 3 \) in the good state that occurs with probability \( \pi = 1/2 \) and pays off \( d^B = 0 \) (nothing) in the bad state that occurs with probability \( 1 - \pi = 1/2 \). Suppose again that this investor has a von Neumann-Morgenstern expected utility function, with a Bernoulli utility function of the constant relative risk aversion (CRRA) form

\[
 u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}.
\]

Finally, recall from our class discussions that the certainty equivalent \( CE(d) \) for the stock is the maximum riskless payoff that the investor is willing to exchange for the stock and the risk premium for the stock \( \Psi(d) \) is the difference between its the expected payoff \( E(d) \) and its certainty equivalent.

a. Assuming that the investor’s constant coefficient of relative risk aversion is \( \gamma = 1/2 \), what is the certainty equivalent \( CE(d) \) for the stock?

b. Assuming instead that the investor’s constant coefficient of relative risk aversion is \( \gamma = 2 \), what is the certainty equivalent \( CE(d) \) for the stock?

c. What is the risk premium for the stock when \( \gamma = 1/2 \)? What is the risk premium when \( \gamma = 2 \)?

5. Risk Aversion and Portfolio Allocation

Consider the portfolio allocation problem faced by an investor who has initial wealth \( Y_0 = 100 \). The investor allocates the amount \( a \) to stocks, which provide return \( r_G = 0.30 \) in a good state that occurs with probability \( 1/2 \) and return \( r_B = 0.05 \) in a bad state that occurs with probability \( 1/2 \). The investor allocates the remaining \( Y_0 - a \) to a risk-free bond, which provides the return \( r_f = 0.10 \) in both states. The investor has von Neumann-Morgenstern expected utility, with Bernoulli utility function of the logarithmic form

\[
 u(Y) = \ln(Y).
\]

a. Write down a mathematical statement of this portfolio allocation problem.

b. Write down the first-order condition for the investor’s optimal choice \( a^* \).

c. Write down the numerical value of the investor’s optimal choice \( a^* \).
1. Stocks, Bonds, and Contingent Claims

a. The stock pays off $d^G = 3$ in the good state and $d^B = 1$ in the bad. The bond pays off one for sure in both states. A contingent claim for the good state pays off one in the good state and zero in the bad. To replicate these payoffs, $s$ and $b$ must satisfy

\[ 3s + b = 1 \]

and

\[ s + b = 0. \]

These two equations are solved with $s = 0.5$ and $b = -0.5$.

b. A contingent claim for the bad state pays off zero in the good state and one in the bad. To replicate these payoffs, $s$ and $b$ must satisfy

\[ 3s + b = 0 \]

and

\[ s + b = 1. \]

These two equations are solved with $s = -0.5$ and $b = 1.5$.

c. Each share of stock sells for $q^s = 1$ at $t = 0$, and each bond sells for $q^b = 0.8$ at $t = 0$. Since the payoffs from the contingent claim for the good state can be replicated by buying 0.5 shares of stock ($s = 0.5$) and selling 0.5 bonds ($b = -0.5$), the claim for the good state should sell for

\[ 1 \times 0.5 + 0.8 \times (-0.5) = 0.5 - 0.4 = 0.1 \]

at $t = 0$. Since the payoffs from the contingent claim for the bad state can be replicated by selling 0.5 shares of stock ($s = -0.5$) and buying 1.5 bonds ($b = 1.5$), the claim for the bad state should sell for

\[ 1 \times (-0.5) + 0.8 \times (1.5) = -0.5 + 1.2 = 0.7 \]

at time $t = 0$. 


2. Concepts of Dominance

a. No. Asset 1 provides a higher return in state 1, while asset 2 provides higher returns in states 2 or 3, so neither exhibits state-by-state dominance over the other.

b. No. The expected return on assets 1 and 2 are

\[
E(\tilde{R}_1) = \frac{1}{3}10 + \frac{1}{3}0 + \frac{1}{3}10 = \frac{20}{3} = 6.67
\]

and

\[
E(\tilde{R}_2) = \frac{1}{3}0 + \frac{1}{3}10 + \frac{1}{3}20 = \frac{30}{3} = 10.
\]

The standard deviations of the returns on assets 1 and 2 are

\[
\sigma(\tilde{R}_1) = \left[\frac{1}{3}(10 - \frac{20}{3})^2 + \frac{1}{3}(0 - \frac{20}{3})^2 + \frac{1}{3}(10 - \frac{20}{3})^2\right]^{1/2}
\]

\[
= \left[\frac{1}{3}(10/3)^2 + \frac{1}{3}(20/3)^2 + \frac{1}{3}(10/3)^2\right]^{1/2}
\]

\[
= \left(\frac{100}{27} + \frac{400}{27} + \frac{100}{27}\right)^{1/2} = \left(\frac{600}{27}\right)^{1/2} = \frac{22.22}{3} = 4.71
\]

and

\[
\sigma(\tilde{R}_2) = \left[\frac{1}{3}(0 - 10)^2 + \frac{1}{3}(10 - 10)^2 + \frac{1}{3}(20 - 10)^2\right]^{1/2}
\]

\[
= \left(\frac{100}{3} + 0 + \frac{100}{3}\right)^{1/2} = \left(\frac{200}{3}\right)^{1/2} = \left(\frac{66.67}{3}\right)^{1/2} = 8.17
\]

Since asset 2 offers a higher expected return but also has a return with a larger standard deviation, neither asset exhibits mean-variance dominance over the other.

c. Asset 1 has the highest Sharpe ratio, since

\[
\frac{E(\tilde{R}_1)}{\sigma(\tilde{R}_1)} = \frac{6.67}{4.71} = 1.42
\]

and

\[
\frac{E(\tilde{R}_2)}{\sigma(\tilde{R}_2)} = \frac{10}{8.17} = 1.22.
\]

3. Expected Utility and Risk Aversion

a. With constant coefficient of relative risk aversion equal to \(\gamma = 1/2\), the investor’s expected utility from the stock is

\[
\left(\frac{1}{2}\right) \left[\frac{(13)^{1/2} - 1}{1/2}\right] + \left(\frac{1}{2}\right) \left[\frac{(10)^{1/2} - 1}{1/2}\right] = 4.77,
\]

whereas the investor’s utility from the bond is

\[
\frac{(11)^{1/2} - 1}{1/2} = 4.63.
\]

Hence, the investor prefers the stock.
b. With constant coefficient of relative risk aversion equal to $\gamma = 2$, the investor’s expected utility from the stock is

$$\left(\frac{1}{2}\right) \left[ \frac{(13)^{-1} - 1}{-1} \right] + \left(\frac{1}{2}\right) \left[ \frac{(10)^{-1} - 1}{-1} \right] = 0.9115,$$

whereas the investor’s utility from the bond is

$$\frac{(11)^{-1} - 1}{1} = 0.9091.$$

Hence, the investor still prefers the stock.

c. Since the Bernoulli utility function in this case is an affine transformation of the Bernoulli utility function in part (a), above, the utility functions represent exactly the same preferences. Although one could confirm this by re-doing the calculations, this observation alone implies that the investor will once again prefer the stock.

4. Certainty Equivalents and Risk Premia

a. With $\gamma = 1/2$, the certainty equivalent for the stock is determined by

$$\frac{[10 + CE(\hat{d})]^{1/2} - 1}{1/2} = \left(\frac{1}{2}\right) \left[ \frac{(13)^{1/2} - 1}{1/2} \right] + \left(\frac{1}{2}\right) \left[ \frac{(10)^{1/2} - 1}{1/2} \right]$$

or, more simply,

$$CE(\hat{d}) = [(1/2)(13)^{1/2} + (1/2)(10)^{1/2}]^2 - 10 = 1.45.$$

b. With $\gamma = 2$ instead, the certainty equivalent is determined by

$$\frac{[10 + CE(\hat{d})]^{-1} - 1}{-1} = \left(\frac{1}{2}\right) \left[ \frac{(13)^{-1} - 1}{-1} \right] + \left(\frac{1}{2}\right) \left[ \frac{(10)^{-1} - 1}{-1} \right]$$

or, more simply,

$$CE(\hat{d}) = [(1/2)(13)^{-1} + (1/2)(10)^{-1}]^{-1} - 10 = 1.30.$$

c. Since the expected payoff from the stock is 1.5, the risk premium when $\gamma = 1/2$ is 0.05 and the risk premium when $\gamma = 2$ is 0.20.

5. Risk Aversion and Portfolio Allocation

a. The investor chooses the amount $a$ allocated to stocks in order to maximize expected utility, which is given by

$$E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\} = (1/2) \ln[110+a(0.30-0.10)]+(1/2) \ln[110+a(0.05-0.10)].$$
b. The first-order condition for the optimal choice $a^*$ is

\[
\left( \frac{1}{2} \right) \left( \frac{0.20}{110 + 0.20a^*} \right) - \left( \frac{1}{2} \right) \left( \frac{0.05}{110 - 0.05a^*} \right) = 0.
\]

c. The solution for $a^*$ can be found from the first-order condition as

\[
0.20(110 - 0.05a^*) = 0.05(110 + 0.20a^*)
\]

\[
22 - 0.01a^* = 5.5 + 0.01a^*
\]

\[
0.02a^* = 16.5
\]

\[
a^* = 825.
\]
1. The Gains From Diversification

Consider portfolios formed from two risky assets, the first with expected return equal to $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 8$ and the second with expected return equal to $\mu_2 = 6$ and standard deviation of its return equal to $\sigma_2 = 6$. Let $w$ denote the fraction of wealth in the portfolio allocated to asset 1 and $1 - w$ the corresponding fraction of wealth allocated to asset 2.

a. Calculate the expected return and the standard deviation of the return on the portfolio that sets $w = 1/2$, assuming that the correlation between the two returns is $\rho_{12} = 0$.

b. Calculate the expected return and the standard deviation of the return on the same portfolio that sets $w = 1/2$, assuming instead that the correlation between the two returns is $\rho_{12} = -0.50$.

c. For which of these two values of $\rho_{12}$ are the gains from diversification larger?

2. Portfolio Allocation with Mean-Variance Utility

Consider an investor with preferences over the mean and variance of the returns on his or her portfolio that are described by the utility function

$$U(\mu_P, \sigma_P) = \mu_P - \left(\frac{A}{2}\right) \sigma_P^2,$$

where $A$ is a parameter that measures the investor’s degree of risk aversion. Suppose that this investor is able to form a portfolio from a risk-free asset with return $r_f$ and a risky asset with expected return equal to $\mu_r$ and variance of its return equal to $\sigma_r^2$.

a. Letting $w$ denote the share of the investor’s wealth allocated to the risky asset and $1 - w$ the share allocated to the risk-free asset, write down an equation for the expected return on his or her portfolio.
b. Next, write down an equation for the variance of the return on his or her portfolio.

c. Write down an equation that shows how the investor’s optimal choice \( w^* \) for the fraction of wealth allocated to the risky asset depends on the risk-free rate \( r_f \), the mean and variance \( \mu_r \) and \( \sigma_r^2 \) of the return on the risky asset, and the parameter \( A \) measuring risk aversion.

3. The CAPM and the APT

Consider an economy in which the random return \( \tilde{r}_i \) on each individual asset \( i \) is determined by the market model

\[
\tilde{r}_i = E(\tilde{r}_i) + \beta_i [\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i,
\]

where, as we discussed in class, \( E(\tilde{r}_i) \) is the expected return on asset \( i \), \( \tilde{r}_M \) is the return on the market portfolio, \( \beta_i \) reflects the covariance between the return on asset \( i \) and the return on the market portfolio, and \( \varepsilon_i \) is an idiosyncratic, firm-specific component. Assume, as Stephen Ross did when developing the arbitrage pricing theory (APT), that there are enough individual assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. Write down the equation, implied by the APT, that links the expected return \( E(\tilde{r}_w) \) on each well-diversified portfolio to the risk-free rate \( r_f \), the expected return \( E(\tilde{r}_M) \) on the market portfolio, and the portfolio’s beta \( \beta_w \).

b. Explain briefly (one or two sentences is all that it should take) how the equation you wrote down to answer part (a), above, differs from the capital asset pricing model’s (CAPM’s) security market line.

c. Suppose you find a well-diversified portfolio with beta \( \beta_w \) that has an expected return that is higher than the expected return given in your answer to part (a), above. In that case, you can buy that portfolio, and sell short a portfolio of equal value that allocates the share \( w \) to the market portfolio and the remaining share to \( 1 - w \) to a risk-free asset. What value of \( w \) will make this trading strategy free of risk, self-financing, but profitable for sure?

4. Optimal Allocations

Consider an economy in which there are two dates, \( t = 0 \) and \( t = 1 \), and two possible states, \( i = 1 \) and \( i = 2 \), at \( t = 1 \), which occur with equal probability, so that \( \pi_1 = \pi_2 = 1/2 \). Suppose, as well, that there are two types of investors, \( j = 1 \) and \( j = 2 \), both of whom have logarithmic Bernoulli utility functions, so that in particular the representative investor of type \( j \) has expected utility

\[
\ln(c_j^0) + \beta[(1/2)\ln(c_j^1) + (1/2)\ln(c_j^2)],
\]

where \( c_j^0 \) denotes this investor’s consumption at \( t = 0 \) and \( c_j^1 \) and \( c_j^2 \) his or her consumption in states \( i = 1 \) and \( i = 2 \) at \( t = 1 \).
Suppose that the aggregate endowment is $w^0 = 30$ at $t = 0$, $w^1 = 30$ in state $i = 1$ at $t = 1$, and $w^2 = 60$ in state $i = 2$ at $t = 1$.

A social planner in this economy divides the aggregate endowment into amounts allocated to representative investors of each type, subject to the resource constraints

$$30 \geq c_0^0 + c_2^0,$$
$$30 \geq c_1^0 + c_1^2,$$
and
$$60 \geq c_1^2 + c_2^2$$
in order to maximize a weighted sum of their expected utilities

$$\theta\{\ln(c_0^0) + \beta[(1/2)\ln(c_1^0) + (1/2)\ln(c_1^2)]\} + (1 - \theta)\{\ln(c_0^1) + \beta[(1/2)\ln(c_1^1) + (1/2)\ln(c_2^1)]\},$$

where the discount factor $\beta = 0.9$.

a. Write down the Lagrangian for the social planner’s problem: choose $c_0^0$, $c_0^2$, $c_1^0$, $c_1^2$, $c_1^1$, and $c_2^2$ to maximize the weighted sum of utilities subject to the three resource constraints.

b. Write down the first-order conditions for the optimal choices of $c_0^0$, $c_0^2$, $c_1^0$, $c_1^2$, $c_1^1$, and $c_2^2$.

c. Suppose that the social planner attaches a weight of $\theta = 2/3$ to the type $j = 1$ investor’s expected utility that is twice as large as the weight $1 - \theta = 1/3$ given to the type $j = 2$ investor’s expected utility. Under this assumption, what are the numerical values of the optimal choices of $c_0^0$, $c_0^2$, $c_1^0$, $c_1^2$, $c_1^1$, and $c_2^2$?

5. Pricing Contingent Claims and Complex Assets Using the Term Structure

Consider a multiperiod economy without uncertainty, where the interest rates on risk-free discount bonds are $r(1) = 0.05$ for a one-year bond and $r(2) = 0.10$ for a two-year bond.

a. Use these data to infer the price of contingent claims $q(1)$ and $q(2)$, where $q(T)$ is the price of the claim that delivers one dollar for sure $T$ years from now.

b. Use your answers from part (a), above, to determine the price today of a two-year coupon bond that makes an interest payment of $10$ at the end of the first year, another interest payment of $10$ at the end of the second year, and a final payment of $100$ also at the end of the second year.

c. Use your answers from part (a), above, to determine the price today of a share of stock that makes no dividend payments, but that can be resold at the price $10$ at the end of the second year.
1. **The Gains From Diversification**

There are two risky assets, the first with expected return equal to $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 8$ and the second with expected return equal to $\mu_2 = 6$ and standard deviation of its return equal to $\sigma_2 = 6$. Let $w$ denote the fraction of wealth in the portfolio allocated to asset 1 and $1 - w$ the corresponding fraction of wealth allocated to asset 2.

a. If $w = 1/2$ and the correlation between the two returns is $\rho_{12} = 0$, the expected return on the portfolio is

$$\mu_P = (1/2)\mu_1 + (1/2)\mu_2 = (1/2) \times 10 + (1/2) \times 6 = 5 + 3 = 8.$$ 

and the standard deviation of the return on the portfolio is

$$\sigma_P = [(1/2)^2 \sigma_1^2 + (1/2)^2 \sigma_2^2]^{1/2} = [(1/4) \times 64 + (1/4) \times 36]^{1/2} = (16 + 9)^{1/2} = 5.$$ 

b. If $w = 1/2$ and the correlation between the two returns is $\rho_{12} = -0.50$, the expected return on the portfolio is still

$$\mu_P = (1/2)\mu_1 + (1/2)\mu_2 = (1/2) \times 10 + (1/2) \times 6 = 5 + 3 = 8.$$ 

but the standard deviation of the return on the portfolio is

$$\sigma_P = [(1/2)^2 \sigma_1^2 + (1/2)^2 \sigma_2^2 + 2(1/2)(1/2)\rho_{12}\sigma_1\sigma_2]^{1/2}$$

$$= [(1/4) \times 64 + (1/4) \times 36 - (1/4) \times 8 \times 6]^{1/2} = 3.6056.$$ 

c. The portfolio described in part (b) has the same expected return but a lower standard deviation of its return than the portfolio in part (a). This shows that the gains from diversification are larger when $\rho_{12} = -0.50$.

2. **Portfolio Allocation with Mean-Variance Utility**

There is a risk-free asset with return $r_f$ and a risky asset with expected return equal to $\mu_r$ and variance of its return equal to $\sigma_r^2$.

a. If an investor forms a portfolio by allocating the share $w$ of his or her wealth to the risky asset and the share $1 - w$ to the risk-free asset, this portfolio has expected return

$$\mu_P = (1 - w)r_f + w\mu_r.$$
b. The variance of the return on the same portfolio is
\[ \sigma_P^2 = w^2 \sigma_r^2. \]

c. With the utility function
\[ U(\mu_P, \sigma_P) = \mu_P - \left( \frac{A}{2} \right) \sigma_P^2, \]
the investor chooses \( w \) to maximize
\[ (1 - w)r_f + w\mu_r - \left( \frac{A}{2} \right) w^2 \sigma_r^2. \]

The first-order condition for this problem is
\[ \mu_r - r_f = Aw^* \sigma_r^2, \]
implying that the optimal choice of \( w \) is
\[ w^* = \frac{\mu_r - r_f}{A\sigma_r^2}. \]

3. The CAPM and the APT

The random return \( \bar{r}_i \) on each individual asset \( i \) is determined by the market model
\[ \bar{r}_i = E(\bar{r}_i) + \beta_i[\bar{r}_M - E(\bar{r}_M)] + \varepsilon_i. \]

a. The APT implies that the expected return on any well-diversified portfolio is given by
\[ E(\bar{r}_w) = r_f + \beta_w[E(\bar{r}_M) - r_f]. \]

b. The relationship from part (a), above, is identical to the CAPM’s security market line, but applies only to well-diversified portfolios and not necessarily to individual assets.

c. If you find a portfolio with an expected return that is higher than what is implied by the equation from part (a), above, you can buy that portfolio and simultaneously sell short a portfolio of equal value that allocates the share \( w = \beta_w \) to the market portfolio and share \( 1 - w = 1 - \beta_w \) to a risk-free asset. This trade will be free of risk and self-financing, but will yield a positive payoff for sure. Arbitrage opportunities like this will not last long in markets where investors can trade in all well-diversified portfolios.

4. Optimal Allocations

A social planner chooses \( c_0^0, c_0^1, c_1^1, c_2^1, c_1^2, c_2^2 \) to maximize a sum of the two investors’ expected utilities
\[ \theta \{ \ln(c_1^0) + 0.9[(1/2) \ln(c_1^1) + (1/2) \ln(c_1^2)] \} + (1 - \theta) \{ \ln(c_2^0) + 0.9[(1/2) \ln(c_2^1) + (1/2) \ln(c_2^2)] \}, \]
subject to the resource constraints
\[ 30 \geq c_1^0 + c_2^0, \]
\[ 30 \geq c_1^1 + c_2^1, \]
and
\[ 60 \geq c_1^2 + c_2^2. \]

a. The Lagrangian for the social planner’s problem is
\[
L = \theta \{ \ln(c_1^0) + 0.9[(1/2) \ln(c_1^1) + (1/2) \ln(c_1^2)] \}
+ (1 - \theta) \{ \ln(c_2^0) + 0.9[(1/2) \ln(c_2^1) + (1/2) \ln(c_2^2)] \}
+ \lambda^0(30 - c_1^0 - c_2^0) + \lambda^1(30 - c_1^1 - c_2^1) + \lambda^2(60 - c_1^2 - c_2^2).
\]

b. The first-order conditions for the social planner’s problem are
\[
\frac{\theta}{c_1^0} = \lambda^0,
\]
\[
\frac{0.9\theta(1/2)}{c_1^1} = \lambda^1,
\]
\[
\frac{0.9\theta(1/2)}{c_2^1} = \lambda^2,
\]
\[
\frac{1 - \theta}{c_2^0} = \lambda^0,
\]
\[
\frac{0.9(1 - \theta)(1/2)}{c_2^1} = \lambda^1,
\]
and
\[
\frac{0.9(1 - \theta)(1/2)}{c_2^2} = \lambda^2,
\]

c. If the social planner attaches a weight of \( \theta = 2/3 \) to the type \( j = 1 \) investor’s expected utility that is twice as large as the weight \( 1 - \theta = 1/3 \) given to the type \( j = 2 \) investor’s expected utility, the first-order conditions from part (b) imply that
\[
\frac{2}{c_1^0} = \frac{1}{c_2^0},
\]
\[
\frac{2}{c_1^1} = \frac{1}{c_2^1},
\]
and
\[
\frac{2}{c_1^2} = \frac{1}{c_2^2}.
\]
In words, the type 1 investor always gets twice as much consumption as the type 2 investor. Using this result, together with the aggregate resource constraints, provides the solutions
\[ c^0_1 = 20 \text{ and } c^0_2 = 10, \]
\[ c^1_1 = 20 \text{ and } c^1_2 = 10, \]
and
\[ c^2_1 = 40 \text{ and } c^2_2 = 20 \]
for the optimal allocations.

5. Pricing Contingent Claims and Complex Assets Using the Term Structure

In a multiperiod economy without uncertainty, the interest rates on risk-free discount bonds are \( r(1) = 0.05 \) for a one-year bond and \( r(2) = 0.10 \) for a two-year bond.

a. In this economy without uncertainty, the two bonds are contingent claims. In particular, the price of a contingent claim that delivers a dollar for sure one year from now sells for the same price
\[ q(1) = P(1) = \frac{1}{1 + r(1)} = 0.9524 \]
as the one-year bond and the price of a contingent claim that delivers one dollar for sure two years from now sells for the same price
\[ q(2) = P(2) = \frac{1}{[1 + r(2)]^2} = 0.8264 \]
as the two-year bond.

b. The answers from part (a), above, imply that the price today of a two-year coupon bond that makes an interest payment of $10 at the end of the first year, another interest payment of $10 at the end of the second year, and a final payment of $100 also at the end of the second year is
\[ 10 \times q(1) + 110 \times q(2) = 9.524 + 82.64 = 92.164. \]

(c). The answers from part (a), above, imply that the price today of a share of stock that makes no dividend payments, but that can be resold for the price $10 at the end of the second year is
\[ 10 \times q(2) = 8.26. \]
1. The Gains From Diversification

Consider portfolios formed from two risky assets, the first with expected return equal to \( \mu_1 = 10 \) and standard deviation of its return equal to \( \sigma_1 = 6 \) and the second with expected return equal to \( \mu_2 = 4 \) and standard deviation of its return equal to \( \sigma_2 = 6 \). Let \( w \) denote the fraction of wealth in the portfolio allocated to asset 1 and \( 1 - w \) the corresponding fraction of wealth allocated to asset 2.

a. Calculate the expected return and the standard deviation of the return on the portfolio that sets \( w = 1/2 \), assuming that the correlation between the two returns is \( \rho_{12} = 0 \).

b. Calculate the expected return and the standard deviation of the return on the same portfolio that sets \( w = 1/2 \), assuming instead that the correlation between the two returns is \( \rho_{12} = -1 \).

c. For which of these two values of \( \rho_{12} \) are the gains from diversification larger?

2. Portfolio Allocation with Mean-Variance Utility

Consider an economy with a risk-free asset with return \( r_f \) and two risky assets, one with random return \( \tilde{r}_1 \) with expected value \( E(\tilde{r}_1) \) and standard deviation \( \sigma_1 \) and the second with random return \( \tilde{r}_2 \) with expected value \( E(\tilde{r}_2) \) and standard deviation \( \sigma_2 \). Let \( \rho_{12} \) denote the correlation between the two random returns on the two risky assets. Suppose that an investor forms a portfolio of these three assets by allocating the share \( w_1 \) of his or her wealth to risky asset 1, with with random return \( \tilde{r}_1 \), share \( w_2 \) to risky asset 2, with random return \( \tilde{r}_2 \), and the remaining share \( 1 - w_1 - w_2 \) to the risk-free asset.

a. Write down a formula for the expected return \( \mu_P \) on this portfolio.

b. Next, write down a formula for the variance \( \sigma^2_P \) of the return on the same portfolio.
c. Suppose that the investor has a preferences defined directly over the mean and variance of the return on his or her portfolio, as described by the utility function

\[ U(\mu_P, \sigma_P) = \mu_P - \left( \frac{A}{2} \right) \sigma_P^2, \]

where \( A \) is a parameter that measures the investor’s degree of risk aversion. Write down the first-order conditions for the investor’s optimal choices \( w_1^* \) and \( w_2^* \) for the shares allocated to each of the two risky assets.

3. A Two-Factor Arbitrage Pricing Theory

Consider an economy in which the random return \( \tilde{r}_i \) on each individual asset \( i \) is given by

\[ \tilde{r}_i = E(\tilde{r}_i) + \beta_{i,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{i,v}[\tilde{r}_V - E(\tilde{r}_V)] + \varepsilon_i \]

where, as we discussed in class, \( E(\tilde{r}_i) \) equals the expected return on asset \( i \), \( \tilde{r}_M \) is the random return on the market portfolio, \( \tilde{r}_V \) is the random return on a “value” portfolio that takes a long position in shares of stock issued by smaller, overlooked companies or companies with high book-to-market values and a corresponding short position is shares of stock issued by larger, more popular companies or companies with low book-to-market values, \( \varepsilon_i \) is an idiosyncratic, firm-specific component, and \( \beta_{i,m} \) and \( \beta_{i,v} \) are the “factor loadings” that measure the extent to which the return on asset \( i \) is correlated with the return on the market and value portfolios. Assume, as Stephen Ross did when developing the arbitrage pricing theory (APT), that there are enough individual assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. Consider, first, a well-diversified portfolio that has \( \beta_{w,m} = \beta_{w,v} = 0 \). Write down the equation, implied by the APT, that links the expected return \( E(\tilde{r}_w) \) on this portfolio to the return \( r_f \) on a portfolio of risk-free assets.

b. Consider, next, two more well-diversified portfolios. portfolio two with \( \beta_{w,m} = 1 \) and \( \beta_{w,v} = 0 \) and portfolio three with \( \beta_{w,m} = 0 \) and \( \beta_{w,v} = 1 \). Write down the equations, implied by this version of the APT, that link the expected returns \( E(\tilde{r}_w^2) \) and \( E(\tilde{r}_w^3) \) on each of these two portfolios to \( r_f, E(\tilde{r}_M), \) and \( E(\tilde{r}_V) \).

c. Suppose you find a fourth well-diversified portfolio that has non-zero values of both \( \beta_{w,m} \) and \( \beta_{w,v} \) and that has expected return

\[ E(\tilde{r}_w^4) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f] + \Delta, \]

where \( \Delta < 0 \) is a negative number. Explain briefly how you could use this portfolio, together with the first three from parts (a) and (b), above, in a trading strategy that involves no risk, requires no money down, but yields a future profit for sure.
4. Consumer Optimization and Contingent Claims Prices

Consider an economy in which there are two dates, $t = 0$ and $t = 1$, and two possible states, $i = 1$ and $i = 2$, at $t = 1$, which occur with equal probability, so that $\pi_1 = \pi_2 = 1/2$. In this economy, a “representative consumer/investor” has expected utility

$$\ln(c^0) + \beta[(1/2)\ln(c^1) + (1/2)\ln(c^2)],$$

where $c^0$ denotes this investor’s consumption at $t = 0$ and $c^1$ and $c^2$ his or her consumption in states $i = 1$ and $i = 2$ at $t = 1$, where the discount factor $\beta = 0.9$.

This investor receives endowments of $w^0$ at $t = 0$ and $w^1$ and $w^2$ in states $i = 1$ and $i = 2$ at $t = 1$. Hence, if we assume as we did in class that the investor trades contingent claims for the two states, his or her budget constraint is

$$w^0 + q^1w^1 + q^2w^2 \geq c^0 + q^1c^1 + q^2c^2,$$

where $q^1$ is the price of the claim for state $i = 1$, $q^2$ is the price of the claim for state $i = 2$, and consumption at $t = 0$ is the numeraire, so that its price is one.

a. Write down the Lagrangian for the investor’s problem: choose $c^0$, $c^1$, and $c^2$ to maximize expected utility subject to the budget constraint.

b. Write down the first-order conditions for the investor’s optimal choices of $c^0$, $c^1$, and $c^2$.

c. Suppose that, as observers of this economy, we see that the consumer chooses $c^0 = 1$, $c^1 = 2$, and $c^2 = 3$. Using this information and the results you obtained in part (b), above, calculate the prices $q^1$ and $q^2$ for the two contingent claims.

5. Prices of Contingent Claims and Complex Assets

Consider an economy in which there are two dates $t = 0$ and $t = 1$, and two possible states, $i = 1$ and $i = 2$, at $t = 1$, which occur with equal probability, so that $\pi_1 = \pi_2 = 1/2$. In this economy, two “complex” assets are traded. The first sells for price $P^0_1 = 2$ at $t = 0$ and makes payoffs of $Z^1_1 = 2$ in state $i = 1$ at $t = 1$ and $Z^2_1 = 4$ in state $i = 2$ at $t = 1$. The second sells for $P^0_2 = 1.25$ at $t = 0$ and makes payoffs $Z^1_2 = 1$ in state $i = 1$ at $t = 1$ and $Z^2_2 = 3$ in state $i = 2$ at $t = 1$.

a. Use this information to compute the price $q^1$ at $t = 0$ of a contingent claim that pays off one in state $i = 1$ at $t = 1$ and zero in state $i = 2$ at $t = 1$ and the price $q^2$ at $t = 0$ of a contingent claim that pays off zero in state $i = 1$ at $t = 1$ and one in state $i = 2$ at $t = 1$.

b. Use your answers from part (a), above, to compute the price $P^0_b$ at $t = 0$ of a bond that pays off $Z^1_b = 1$ in state $i = 1$ at $t = 1$ and $Z^2_b = 1$ in state $i = 2$ at $t = 1$.

c. Use your answers from part (a), above, to compute the price $P^0_s$ at $t = 0$ of a share of stock that makes no dividend payments, but can be sold for $P^1_s = 2$ in state $i = 1$ at $t = 1$ and $P^2_s = 3$ in state $i = 2$ at $t = 1$.  

1. The Gains From Diversification

There are two risky assets, the first with expected return equal to $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 6$ and the second with expected return equal to $\mu_2 = 4$ and standard deviation of its return equal to $\sigma_2 = 6$. Let $w$ denote the fraction of wealth in the portfolio allocated to asset 1 and $1 - w$ the corresponding fraction of wealth allocated to asset 2.

a. If $w = 1/2$ and the correlation between the two returns is $\rho_{12} = 0$, the expected return on the portfolio is

$$\mu_p = (1/2)\mu_1 + (1/2)\mu_2 = (1/2) \times 10 + (1/2) \times 4 = 5 + 2 = 7.$$ 

and the standard deviation of the return on the portfolio is

$$\sigma_p = \sqrt{(1/2)^2 \sigma_1^2 + (1/2)^2 \sigma_2^2} = \sqrt{(1/4) \times 36 + (1/4) \times 36} = \sqrt{9} = 3.$$ 

b. If $w = 1/2$ and the correlation between the two returns is $\rho_{12} = -1$, the expected return on the portfolio is still

$$\mu_p = (1/2)\mu_1 + (1/2)\mu_2 = (1/2) \times 10 + (1/2) \times 4 = 5 + 2 = 7.$$ 

but the standard deviation of the return on the portfolio is

$$\sigma_p = \sqrt{(1/2)^2 \sigma_1^2 + (1/2)^2 \sigma_2^2 + 2(1/2)(1/2)\rho_{12}\sigma_1\sigma_2} = \sqrt{(1/4) \times 36 + (1/4) \times 36 - (1/2) \times 6 \times 6} = \sqrt{9} = 3.$$ 

c. The portfolio described in part (b) has the same expected return but a lower standard deviation of its return than the portfolio in part (a): in fact, the portfolio in (b) is risk free. This shows that the gains from diversification are larger when $\rho_{12} = -1$.

2. Portfolio Allocation with Mean-Variance Utility

There is a risk-free asset with return $r_f$ and two risky assets, one with random return $\tilde{r}_1$ with expected value $E(\tilde{r}_1)$ and standard deviation $\sigma_1$ and the second with random return $\tilde{r}_2$ with expected value $E(\tilde{r}_2)$ and standard deviation $\sigma_2$. Let $\rho_{12}$ denote the correlation between the two random returns on the two risky assets. The investor forms a portfolio of these three assets by allocating the share $w_1$ of his or her wealth to risky asset 1, with with random return $\tilde{r}_1$, share $w_2$ to risky asset 2, with random return $\tilde{r}_2$, and the remaining share $1 - w_1 - w_2$ to the risk-free asset.
a. The expected return on this portfolio is
\[ \mu_P = (1 - w_1 - w_2)r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2). \]

b. The variance of the return of the portfolio is
\[ \sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2. \]

c. If the investor has preferences described by the utility function
\[ U(\mu_P, \sigma_P) = \mu_P - \left( \frac{A}{2} \right) \sigma_P^2, \]
then he or she chooses \( w_1 \) and \( w_2 \) to maximize
\[ (1 - w_1 - w_2)r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2) - \left( \frac{A}{2} \right) (w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2). \]

The first-order conditions for the investor’s optimal choices \( w_1^* \) and \( w_2^* \) are
\[ E(\tilde{r}_1) - r_f = A(w_1^* \sigma_1^2 + w_2^* \rho_{12}\sigma_1\sigma_2) \]
and
\[ E(\tilde{r}_2) - r_f = A(w_2^* \sigma_2^2 + w_1^* \rho_{12}\sigma_1\sigma_2). \]

3. A Two-Factor Arbitrage Pricing Theory

The random return \( \tilde{r}_i \) on each individual asset \( i \) is given by
\[ \tilde{r}_i = E(\tilde{r}_i) + \beta_{i,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{i,v}[\tilde{r}_V - E(\tilde{r}_V)] + \varepsilon_i. \]

a. The APT implies that a well-diversified portfolio with \( \beta_{w,m} = \beta_{w,v} = 0 \) has an expected return equal to the risk-free rate:
\[ E(\tilde{r}_w^1) = r_f \]

b. The APT implies that portfolio two, with \( \beta_{w,m} = 1 \) and \( \beta_{w,v} = 0 \), has expected return that coincides with the expected return on the market portfolio,
\[ E(\tilde{r}_w^2) = E(\tilde{r}_M), \]
while portfolio three, with \( \beta_{w,m} = 0 \) and \( \beta_{w,v} = 1 \), has expected return that coincides with the expected return on the “value portfolio,”
\[ E(\tilde{r}_w^3) = E(\tilde{r}_V). \]
c. If you find a fourth well-diversified portfolio that has non-zero values of both $\beta_{w,m}$ and $\beta_{w,v}$ and that has expected return

$$E(\tilde{r}_w^4) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f] + \Delta,$$

where $\Delta < 0$ is a negative number, you should form a fifth portfolio that allocates the share $\beta_{w,m}$ to the market portfolio, share $\beta_{w,v}$ to the value portfolio, and the remaining share $1 - \beta_{w,m} - \beta_{w,v}$ to risk-free assets. This fifth portfolio will have expected return

$$E(\tilde{r}_w^5) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f],$$

which is higher than the expected return on portfolio 4. But since portfolios 4 and 5 have the same values of both $\beta_{w,m}$ and $\beta_{w,v}$, a strategy of buying portfolio 5 and selling short an equal dollar amount of portfolio 4 will be free of risk, requires no money down, but yields a future profit for sure.

4. Consumer Optimization and Contingent Claims Prices

The representative consumer/investor chooses $c^0$, $c^1$, and $c^2$ to maximize the expected utility function

$$\ln(c^0) + 0.9[(1/2) \ln(c^1) + (1/2) \ln(c^2)],$$

subject to the budget constraint

$$w^0 + q^1w^1 + q^2w^2 \geq c^0 + q^1c^1 + q^2c^2,$$

where $q^1$ is the price of the claim for state $i = 1$, $q^2$ is the price of the claim for state $i = 2$, and consumption at $t = 0$ is the numeraire, so that its price is one.

a. The Lagrangian for the investor’s problem is

$$L = \ln(c^0) + 0.9[(1/2) \ln(c^1) + (1/2) \ln(c^2)] + \lambda(w^0 + q^1w^1 + q^2w^2 - c^0 - q^1c^1 - q^2c^2).$$

b. The first-order conditions for the investor’s optimal choices are

$$\frac{1}{c^0} - \lambda = 0,$$

$$\frac{0.9(1/2)}{c^1} - \lambda q^1 = 0,$$

and

$$\frac{0.9(1/2)}{c^2} - \lambda q^2 = 0.$$ 

c. The first-order conditions from part (b), above, link the investor’s optimal choices to the contingent claims prices via

$$q^1 = \frac{0.9(1/2)c^0}{c^1} \text{ and } q^2 = \frac{0.9(1/2)c^0}{c^2}. $$
Therefore, if we, as observers of this economy, see that the consumer chooses \( c^0 = 1, c^1 = 2, \) and \( c^2 = 3, \) we can infer that the contingent claims prices are

\[
q^1 = \frac{0.9(1/2)}{2} = \frac{0.9}{4} = 0.225 \text{ and } q^2 = \frac{0.9(1/2)}{3} = 0.15.
\]

5. Prices of Contingent Claims and Complex Assets

Two complex assets are traded. The first sells for price \( P^0_1 = 2 \) at \( t = 0 \) and makes payoffs of \( Z^1_1 = 2 \) in state \( i = 1 \) at \( t = 1 \) and \( Z^1_2 = 4 \) in state \( i = 2 \) at \( t = 1. \) The second sells for \( P^0_2 = 1.25 \) at \( t = 0 \) and makes payoffs \( Z^2_1 = 1 \) in state \( i = 1 \) at \( t = 1 \) and \( Z^2_2 = 3 \) in state \( i = 2 \) at \( t = 1. \)

a. Consider a portfolio that consists of \( w^1_1 \) units of asset 1 and \( w^1_2 \) units of asset 2. If we want this portfolio to replicate the payoffs made by a contingent claim for state 1, its payoffs must satisfy

\[
2w^1_1 + w^2_1 = 1
\]

and

\[
4w^1_1 + 3w^2_1 = 0.
\]

These equations both hold when \( w^1_1 = 3/2 \) and \( w^2_1 = -2. \) Since the cost of assembling this portfolio must equal the price of the contingent claim, these numbers imply that

\[
q^1 = (3/2) \times 2 - 2 \times 1.25 = 0.5.
\]

Next, consider a portfolio that consists of \( w^2_1 \) units of asset 1 and \( w^2_2 \) units of asset 2. If we want this portfolio to replicate the payoffs made by a contingent claim for state 2, its payoffs must satisfy

\[
2w^1_2 + w^2_2 = 0
\]

and

\[
4w^1_2 + 3w^2_2 = 1.
\]

These equations both hold when \( w^1_2 = -1/2 \) and \( w^2_1 = 1. \) Since the cost of assembling this portfolio must equal the price of the contingent claim, these numbers imply that

\[
q^2 = (-1/2) \times 2 + 1 \times 1.25 = 0.25.
\]

b. Since the payoffs on the bond can be replicated by buying one contingent claim for each of the two states, the bond price must equal

\[
P^0_b = q^1 + q^2 = 0.75.
\]

c. Since the payoffs from the stock can be replicated by buying two contingent claims for state 1 and three contingent claims for state 2, the stock price must equal

\[
P^0_s = 2 \times q^1 + 3 \times q^2 = 1.75.
\]
This exam has five questions on four pages; before you begin, please check to make sure that your copy has all five questions and all four pages. The five questions will be weighted equally in determining your overall exam score.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

1. Optimal Saving

Consider a consumer who receives income of $Y$ at the beginning of period $t = 0$, which he or she divides up into an amount $c_0$ to be consumed and an amount $s$ to be saved, subject to

$$Y \geq c_0 + s.$$ 

Suppose that the consumer receives no additional income in period $t = 1$, so that all of his or her consumption $c_1$ during that period has to be purchased with the savings and the interest earned on savings from period $t = 0$. Letting $r$ denote the interest rate, this means that

$$(1 + r)s \geq c_1.$$ 

As in class, we can combine these two single-period constraints to obtain the consumer’s “lifetime” budget constraint

$$Y \geq c_0 + \frac{c_1}{1 + r}.$$

Suppose, further, that the consumer’s preferences over consumption during the two periods are described by the utility function

$$\ln(c_0) + \beta \ln(c_1),$$

where $\ln$ denotes the natural logarithm and $\beta$, satisfying $0 < \beta < 1$, determines how patient ($\beta$ larger) or impatient ($\beta$ smaller) the consumer is.

a. Set up the Lagrangian for this consumer’s problem: choose $c_0$ and $c_1$ to maximize the utility function subject to the lifetime budget constraint. Then write down the two first-order conditions that characterize the consumer’s optimal choices $c_0^*$ and $c_1^*$ of consumption in each of the two periods.

b. Suppose now that, by coincidence, the interest rate $r$ and the discount factor $\beta$ are related so that

$$\beta(1 + r) = 1$$
or, equivalently,

\[ \beta = \frac{1}{1 + r}. \]

When this relation linking \( \beta \) and \( 1 + r \) holds, what do the first-order conditions you derived in part (a), above, imply about the consumer’s optimal choice of consumption growth \( c_1^*/c_0^* \)? Is consumption rising, falling, or staying constant over time?

c. Suppose, finally, that the consumer’s income during period \( t = 0 \) is \( Y = 210 \) and that the interest rate is ten percent, or \( r = 0.10 \). Still assuming that the relation linking \( \beta \) and \( 1 + r \) given in part (b), above, holds, what are the numerical values for the consumer’s optimal choices \( c_0^* \) and \( c_1^* \)? \textit{Note:} to find these numerical solutions, you may also have to use the consumer’s lifetime budget constraint, which will hold as the equality

\[ Y = c_0^* + \frac{c_1^*}{1 + r} \]

when consumptions are chosen optimally.

2. Pricing Risk-Free Assets

Assume, for all the examples in this question, there that is no uncertainty: all specified payments made by all assets will be received for sure. Suppose, in particular, that the interest rate on one-year, risk-free discount bonds is 5 percent \((r_1 = 0.05)\) and the annualized interest rate on two-year, risk-free discount bonds is 10 percent \((r_2 = 0.10)\).

a. What is the price of a one-year, risk-free discount bond with face (or par) value equal to \$1000? What is the price of a two-year, risk-free discount bond with face value equal to \$1000?

b. What is the price of a two-year coupon bond that makes an annual interest payment of \$100 each year for the next two years then returns face value \$1000 at the end of the second year?

c. What is the price of a risk-free asset that pays off \$1 at the end of the first year and \$100 at the end of the second year? What is the price of a risk-free asset that pays off \$100 at the end of the first year and \$1 at the end of the second year?
3. Risk Aversion, Certainty Equivalents, and Risk Premia

Consider a risk-averse investor with von Neumann-Morgenstern expected utility and a Bernoulli utility function of the constant relative risk aversion form

\[ u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}, \]

with \( \gamma = 1/2 \). Suppose that this investor has initial income \( Y_0 = 100 \) and has the chance to acquire a risky asset that pays off \( Z^G = 75 \) in a good state that occurs with probability \( \pi = 0.5 \) but pays off only \( Z^B = 25 \) in a bad state that occurs with probability \( 1 - \pi = 0.5 \).

a. Recall from our discussions in class that the certainty equivalent \( CE(\tilde{Z}) \) associated with this risky asset is the maximum riskless payoff that the investor would be willing to give up in order to acquire the risky asset. What is the numerical value for the certainty equivalent \( CE(\tilde{Z}) \)?

b. Recall also from our discussions in class that the risk premium \( \Psi(\tilde{Z}) \) associated with the risky asset is the difference between the expected payoff \( E(\tilde{Z}) \) from the risky asset and the certainty equivalent \( CE(\tilde{Z}) \). What is the numerical value for the risk premium \( \Psi(\tilde{Z}) \)?

c. Suppose that instead of having \( \gamma = 1/2 \), the investor had a coefficient of relative risk aversion equal to \( \gamma = 5 \). Would the certainty equivalent in this case with \( \gamma = 5 \) be larger or smaller than it was in part (a) above with \( \gamma = 1/2 \)? Would the risk premium in this case with \( \gamma = 5 \) be larger or smaller than it was in part (a) above with \( \gamma = 1/2 \)?

Note: to answer these two question in part (c), you don’t necessarily have to compute the numerical values of the certainty equivalent and risk premium when \( \gamma = 5 \); all you need to do is to indicate whether each is larger or smaller than in parts (a) and (b), when \( \gamma = 1/2 \) instead.
4. Insurance

Consider a consumer with income $100 who faces a 50 percent probability of suffering a loss that reduces his or her income to $50. Suppose that this consumer can buy an insurance policy for $x$ that protects him or her fully against this loss by paying him or her $50 to make up for the loss if it occurs. Finally, assume that the consumer has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the constant relative risk aversion form

\[ u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}, \]

with $\gamma = 2$.

a. Write down an expression for the consumer’s expected utility if he or she decides to buy the insurance.

b. Write down an expression for the consumer’s expected utility if he or she decides not to buy the insurance.

c. Suppose, more specifically, that the insurance policy costs $20. If the consumer maximizes expected utility, will he or she buy the insurance policy?

5. Risk Aversion and Investment Decisions

Consider an investor with initial wealth equal to $Y_0 = 100$, who splits that wealth up into an amount $a$ allocated to the stock market and an amount $Y_0 - a$ allocated to a bank account. The return $\tilde{r}$ on funds invested in the stock market is random: in a good state, which occurs with probability $\pi = 0.5$, it equals $r^G = 0.12$ (twelve percent), and in a bad state, with occurs with probability $1 - \pi = 0.5$, it equals $r^B = -0.01$ (minus one percent). The return $r_f$ on funds deposited in the bank account is not random: it equals $r_f = 0.05$ (five percent) for sure. The investor has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the logarithmic form

\[ u(Y) = \ln(Y). \]

a. Write down a mathematical statement of this portfolio allocation problem.

b. Write down the first-order condition for the investor’s optimal choice $a^*$. 

c. Write down the numerical value of the investor’s optimal choice $a^*$.
1. Optimal Saving

The consumer chooses $c_0$ and $c_1$ to maximize the utility function

$$\ln(c_0) + \beta \ln(c_1),$$

subject to the lifetime budget constraint

$$Y \geq c_0 + \frac{c_1}{1 + r}.$$

a. The Lagrangian for the consumer’s problem is

$$L = \ln(c_0) + \beta \ln(c_1) + \lambda \left( Y - c_0 - \frac{c_1}{1 + r} \right).$$

The first-order condition for $c_0$ is

$$\frac{1}{c_0^*} - \lambda^* = 0$$

and the first-order condition for $c_1$ is

$$\frac{\beta}{c_1^*} - \frac{\lambda^*}{1 + r} = 0.$$

b. When

$$\beta = \frac{1}{1 + r},$$

the first-order condition for $c_1$ becomes

$$\frac{1}{c_1^*} - \lambda^* = 0.$$

Comparing this last equation to the first-order condition for $c_0$ reveals that in this case

$$c_0^* = c_1^*,$$

or

$$c_1^*/c_0^* = 1,$$

so that consumption is staying constant over time.
c. When $Y = 210$, $r = 0.10$, and $c_0^* = c_1^*$, the consumer’s lifetime budget constraint

$$Y = c_0^* + \frac{c_1^*}{1 + r}$$
implies that

$$210 = c_0^* + \frac{c_0^*}{1.10} = c_0^* \left(1 + \frac{1}{1.10}\right) = c_0^* \left(\frac{2.10}{1.10}\right)$$

and hence

$$c_0^* = c_1^* = 110.$$ 

In period $t = 0$, the consumer spends 110 and saves 100. With the 10 percent interest rate, this lets the consumer spend 110 during period $t = 1$ as well.

## 2. Pricing Risk-Free Assets

The interest rate on one-year, risk-free discount bonds is $r_1 = 0.05$ and the annualized interest rate on two-year, risk-free discount bonds is $r_2 = 0.10$.

a. The price of a one-year, risk-free discount bond with face (or par) value equal to $1000$ is

$$P_1 = \frac{1000}{1.05} = 952.38.$$  

The price of a two-year, risk-free discount bond with face value equal to $1000$ is

$$P_2 = \frac{1000}{1.10^2} = 826.45.$$ 

b. The price of a two-year coupon bond that makes an annual interest payment of $100$ each year for the next two years then returns face value $1000$ at the end of the second year is

$$P^C_2 = \frac{100}{1.05} + \frac{100}{1.10^2} + \frac{1000}{1.10^2} = 1004.33.$$ 

c. The price of a risk-free asset that pays off $1$ at the end of the first year and $100$ at the end of the second year is

$$P^A_1 = \frac{1}{1.05} + \frac{100}{1.10^2} = 83.60.$$ 

The price of a risk-free asset that pays off $100$ at the end of the first year and $1$ at the end of the second year is

$$P^A_2 = \frac{100}{1.05} + \frac{1}{1.10^2} = 96.06.$$
3. Risk Aversion, Certainty Equivalents, and Risk Premia

The investor has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the constant relative risk aversion form

\[ u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}, \]

with \(\gamma = 1/2\). This investor has initial income \(Y_0 = 100\), and has the chance to acquire a risky asset, that pays off \(Z^G = 75\) in a good state that occurs with probability \(\pi = 0.5\) but pays off only \(Z^B = 25\) in a bad state that occurs with probability \(1 - \pi = 0.5\).

a. The certainty equivalent \(CE(\tilde{Z})\) associated with this risky asset is the maximum riskless payoff that the investor would be willing to give up in order to acquire the risky asset. To find the certainty equivalent, set the utility from getting the \(CE(\tilde{Z})\) for sure to the expected utility from acquiring the risky asset instead:

\[ (100 + CE(\tilde{Z}))^{1/2} = 0.5 \left( \frac{175^{1/2}}{1/2} \right) + 0.5 \left( \frac{125^{1/2}}{1/2} \right). \]

Dividing through by 1/2 and squaring both sides yields

\[ 100 + CE(\tilde{Z}) = 148.95. \]

Therefore,

\[ CE(\tilde{Z}) = 48.95. \]

b. The risk premium \(\Psi(\tilde{Z})\) associated with the risky asset is the difference between the expected payoff \(E(\tilde{Z})\) from the risky asset and the certainty equivalent \(CE(\tilde{Z})\). Since the risky asset has

\[ E(\tilde{Z}) = 0.5(75) + 0.5(25) = 50, \]

the risk premium is

\[ \Psi(\tilde{Z}) = 50 - 48.95 = 1.05. \]

c. If instead of having \(\gamma = 1/2\), the investor had a coefficient of relative risk aversion equal to \(\gamma = 5\), he or she would be more risk averse. The certainty equivalent would be lower and the risk premium would be larger than they were in part (a) above.

4. Insurance

The consumer has income $100 and faces a 50 percent probability of suffering a loss that reduces his or her income to $50. The consumer can buy an insurance policy for $x that protects him or her fully against this loss by paying him or her $50 to make up for this loss if it occurs, and the consumer has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the constant relative risk aversion form

\[ u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}, \]

with \(\gamma = 2\).
a. If the consumer decides to buy the insurance, his or her expected utility is

\[ \frac{(100 - x)^{-1} - 1}{-1} = 1 - \frac{1}{100 - x}. \]

b. If the consumer decides not to buy the insurance, his or her expected utility is

\[ 0.5 \left( \frac{100^{-1} - 1}{-1} \right) + 0.5 \left( \frac{50^{-1} - 1}{-1} \right) = 1 - 0.5 \left( \frac{1}{100} + \frac{1}{50} \right) = 0.985. \]

c. The insurance policy costs $20, and the expected loss is $25, so we know in advance that any risk-averse investor will buy the insurance. But this outcome can be confirmed by noting that with \( x = 20 \), the solution from part (a), above, implies that the consumer’s expected utility when buying insurance is

\[ 1 - \frac{1}{80} = 0.9875, \]

which is larger than expected utility when not buying insurance given by the solution to part (b), above.

5. Risk Aversion and Investment Decisions

The investor has initial wealth equal to \( Y_0 = 100 \), and splits that wealth up into an amount \( a \) allocated to the stock market and an amount \( Y_0 - a \) allocated to a bank account. The return \( \tilde{r} \) on funds invested in the stock market is random: in a good state, which occurs with probability \( 1/2 \), it equals \( r^G = 0.12 \) (ten percent), and in a bad state, with occurs with probability \( 1/2 \), it equals \( r^B = -0.01 \) (minus one percent). The return \( r_f \) on funds deposited in the bank account is not random: it equals \( r_f = 0.05 \) (four precent) for sure. The investor has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the logarithmic form

\[ u(Y) = \ln(Y). \]

a. In general, the investor’s problem can be expressed mathematically as

\[ \max_a E\{u[(1 + r_f)Y_0 + a(\tilde{r} - r_f)]\}, \]

but given the assumptions about the utility function and the asset returns made above, the problem can be written more specifically here as

\[ \max_a 0.5 \ln(105 + 0.07a) + 0.5 \ln(105 - 0.06a). \]

b. The first-order condition for the investor’s optimal choice \( a^* \) is

\[ \frac{0.5(0.07)}{105 + 0.07a^*} - \frac{0.5(0.06)}{105 - 0.06a^*} = 0. \]
c. The first-order condition for $a^*$ from part (b), above, implies that

$$\frac{0.07}{105 + 0.07a^*} = \frac{0.06}{105 - 0.06a^*}$$

$$0.07(105 - 0.06a^*) = 0.06(105 + 0.07a^*)$$

$$(0.07 - 0.06)105 = 2(0.06)(0.07)a^*$$

$$a^* = \frac{1.05}{2(0.06)(0.07)} = 125.$$
Final Exam

EC379.01 - Financial Economics  
Boston College, Department of Economics  
Peter Ireland  
Spring 2014

Thursday, May 1, 10:30 - 11:45am

This exam has four questions on three pages; before you begin, please check to make sure that your copy has all four questions and all three pages. The four questions will be weighted equally in determining your overall exam score.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

1. Portfolio Allocation and the Gains from Diversification

Consider portfolios formed from two risky assets, the first with expected return equal to \( \mu_1 = 10 \) and standard deviation of its return equal to \( \sigma_1 = 2 \) and the second with expected return equal to \( \mu_2 = 5 \) and standard deviation of its return equal to \( \sigma_2 = 1 \). Let \( w \) denote the fraction of wealth in the portfolio allocated to asset 1 and \( 1 - w \) the corresponding fraction of wealth allocated to asset 2.

a. Calculate the expected return and the standard deviation of the return on the portfolio that sets \( w = 1/2 \), assuming that the correlation between the two returns is \( \rho_{12} = 0 \).

b. Calculate the expected return and the standard deviation of the return on the same portfolio that sets \( w = 1/2 \), assuming instead that the correlation between the two returns is \( \rho_{12} = -0.25 \).

c. Finally, suppose the correlation between the two returns is \( \rho_{12} = -1 \). For what value of \( w \) will the portfolio of the two assets be entirely risk free, in the sense that the standard deviation of the return on the portfolio equals zero?
2. The Capital Asset Pricing Model

Suppose that the random return $\tilde{r}_M$ on the market portfolio has expected value $E(\tilde{r}_M) = 0.07$ and variance $\sigma^2_M = 0.025$ and that the return on risk-free assets is $r_f = 0.01$.

a. According to the capital asset pricing model, what is the expected return on a risky asset with random return $\tilde{r}_j$ that has a covariance $\sigma_{jM} = 0$ of zero with the random return on the market?

b. According to the capital asset pricing model, what is the expected return on the risky asset if, instead, its random return $\tilde{r}_j$ has a covariance of $\sigma_{jM} = 0.025$ with the random return on the market?

c. According to the capital asset pricing model, what is the expected return on the risky asset if, instead, its random return $\tilde{r}_j$ has a covariance of $\sigma_{jM} = 0.050$ with the random return on the market?

3. The Market Model and Arbitrage Pricing Theory

Consider a version of the arbitrage pricing theory that is built on the assumption that the random return $\tilde{r}_i$ on each individual asset $i$ is determined by the market model

$$\tilde{r}_i = E(\tilde{r}_i) + \beta_i[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i$$

where, as we discussed in class, $E(\tilde{r}_i)$ is the expected return on asset $i$, $\tilde{r}_M$ is the return on the market portfolio and $E(\tilde{r}_M)$ is the expected return on the market portfolio, $\beta_i$ is the same beta for asset $i$ as in the capital asset pricing model, and $\varepsilon_i$ is an idiosyncratic, firm-specific component. Assume, as Stephen Ross did when developing the APT, that there are enough assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. Write down the equation, implied by the APT, for the random return $\tilde{r}_w$ on a well-diversified portfolio with beta $\beta_w$.

b. Write down the equation, implied by the APT, for the expected return $E(\tilde{r}_w)$ on this well-diversified portfolio with beta $\beta_w$.

c. Suppose that you find another well-diversified portfolio with the same beta $\beta_w$ that has an expected return that is lower than the expected return given in your answer to part (b), above. Describe briefly (a sentence or two is all that it should take) the trading opportunity provided by this discrepancy that is free of risk, self-financing, but profitable for sure.
4. Stocks, Options, and Contingent Claims

Consider a version of the no-arbitrage variant of the Arrow-Debreu pricing model in which there are two periods: \( t = 0 \), when investment decisions are made, and \( t = 1 \), when investment payoffs become known and are received. Looking ahead from period \( t = 0 \), there are three possible states, \( i = 1, 2, 3 \), during period \( t = 1 \). Markets are complete, in that contingent claims are traded at \( t = 0 \) for all three states at \( t = 1 \). Suppose, in particular, that at \( t = 0 \), a contingent claim that pays off one dollar in state \( i = 1 \) at \( t = 1 \) and zero otherwise sells for \( q^1 = 0.60 \). Suppose, similarly, that a contingent claim for state \( i = 2 \) at \( t = 1 \) sells for \( q^2 = 0.20 \) at \( t = 0 \) and that a contingent claim for state \( i = 3 \) at \( t = 1 \) sells for \( q^3 = 0.15 \) at \( t = 0 \).

a. Suppose that, in addition to the three contingent claims, another asset – a stock – trades at \( t = 0 \). Suppose that the price of a share of stock at \( t = 1 \) depends on the state: it can be sold for \( P^1_s = 1 \) in state \( i = 1 \) at \( t = 1 \), \( P^2_s = 2 \) in state \( i = 2 \) at \( t = 1 \), and \( P^3_s = 3 \) in state \( i = 3 \) at \( t = 1 \); therefore, \( P^1_s, P^2_s, \) and \( P^3_s \) measure the payoffs in each state at \( t = 1 \) from investing in one share of stock at \( t = 0 \). Given the contingent claims prices, what is the numerical value of the price \( P^0_s \) at which the stock should trade at \( t = 0 \)?

b. Now consider a call option that gives the owner the right, but not the obligation, to purchase one share of stock at \( t = 1 \) at the strike price \( K = 2 \). Let \( C^1, C^2, \) and \( C^3 \) denote the payoffs that an investor who purchases this call option at \( t = 0 \) receives in each state of the world \( i = 1, 2, 3 \) at \( t = 1 \). What are the numerical values of \( C^1, C^2, \) and \( C^3 \)?

c. At what price \( V_0 \) will the call option described in part (b), above, sell for at \( t = 0 \)?
1. Portfolio Allocation and the Gains from Diversification

The first asset has expected return equal to $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 2$ and the second asset has expected return equal to $\mu_2 = 5$ and standard deviation of its return equal to $\sigma_2 = 1$. In each portfolio, $w$ denotes the fraction of wealth in the portfolio allocated to asset 1 and $1 - w$ the corresponding fraction of wealth allocated to asset 2. In general, the formulas for the expected return and the standard deviation of the return of these portfolios are given by

$$\mu_w = w\mu_1 + (1 - w)\mu_2$$

and

$$\sigma_w = \sqrt{w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}}^{1/2},$$

where $\rho_{12}$ is the correlation between the two individual assets’ returns.

a. Assuming that $\rho_{12} = 0$, the expected return on the portfolio that sets $w = 1/2$ is

$$(1/2)\mu_1 + (1/2)\mu_2 = 7.5$$

and the standard deviation of this portfolio’s random return is

$$[\sqrt{(1/2)^2\sigma_1^2 + (1/2)^2\sigma_2^2}]^{1/2} = 1.25^{1/2} = 1.118.$$  

b. Assuming instead that $\rho_{12} = -0.25$, the expected return on the portfolio that sets $w = 1/2$ is still

$$(1/2)\mu_1 + (1/2)\mu_2 = 7.5$$

but the standard deviation of this portfolio’s random return falls to

$$[\sqrt{(1/2)^2\sigma_1^2 + (1/2)^2\sigma_2^2 - 2(1/2)(1/2)\sigma_1\sigma_2\rho_{12}}]^{1/2} = 1^{1/2} = 1.$$  

c. If the correlation between the two returns is $\rho_{12} = -1$, the standard deviation of the portfolio’s return is

$$\sigma_w = \sqrt{w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 - 2w(1 - w)\sigma_1\sigma_2}^{1/2} = \sqrt{4w^2 + (1 - w)^2 - 4w(1 - w)}^{1/2}.$$  

Therefore, to make this portfolio risk free, we need to find the value of $w$ that makes

$$4w^2 + (1 - w)^2 - 4w(1 - w) = 0.$$
Since \((1 - w)^2 = 1 - 2w + w^2\) and \(4w(1 - w) = 4w - 4w^2\), this last equation requires that
\[
4w^2 + 1 - 2w + w^2 - 4w + 4w^2 = 0
\]
\[
9w^2 - 6w + 1 = 0
\]
or
\[
(3w - 1)^2 = 0.
\]
But this last equation can only hold if \(w = 1/3\). Therefore, risk-free portfolio has one-third of its funds allocated to asset one and two-thirds allocated to asset two.

2. The Capital Asset Pricing Model

The random return \(\tilde{r}_M\) on the market portfolio has expected value \(E(\tilde{r}_M) = 0.07\) and variance \(\sigma^2_M = 0.025\) and the return on risk-free assets is \(r_f = 0.01\). In general, the capital asset pricing model implies that the expected return on any individual asset \(j\) is
\[
E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]
\]
where the asset’s beta
\[
\beta_j = \frac{\sigma_{jM}}{\sigma^2_M}
\]
depends on the covariance \(\sigma_{jM}\) between its random return \(\tilde{r}_j\) and the random return \(\tilde{r}_M\) on the market portfolio.

a. Since an asset with \(\sigma_{jM} = 0\) has a beta of zero, its expected return equals the risk-free rate \(r_f = 0.01\).

b. Since an asset with \(\sigma_{jM} = 0.025\) has a beta of one, its expected return equals the expected return \(E(\tilde{r}_M) = 0.07\) on the market portfolio.

c. Since an asset with \(\sigma_{jM} = 0.050\) has a beta of two, its expected return is
\[
E(\tilde{r}_j) = r_f + 2[E(\tilde{r}_M) - r_f] = r_f + 2E(\tilde{r}_M) - 2r_f = 2E(\tilde{r}_M) - r_f = 0.14 - 0.01 = 0.13.
\]

3. Arbitrage Pricing Theory

In this version of the arbitrage pricing theory the random return \(\tilde{r}_i\) on each individual asset \(i\) is determined by the market model
\[
\tilde{r}_i = E(\tilde{r}_i) + \beta_i[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i.
\]

a. Since the defining characteristic of a well-diversified portfolio is that it has no idiosyncratic risk, the random return \(\tilde{r}_w\) on a well-diversified portfolio with beta \(\beta_w\) is
\[
\tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)].
\]
b. Since the random return on this well-diversified portfolio can be replicated perfectly by another well diversified portfolio that allocates the share $\beta_w$ of its funds to the market portfolio and the remaining share $1 - \beta_w$ to a portfolio of risk-free assets, the absence of arbitrage opportunities requires that

$$E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f].$$

c. If you find another well-diversified portfolio with the same beta $\beta_w$ that has an expected return that is lower than the expected return given in the answer to part (b), above, you can take a long position worth $x$ in the portfolio from part (b) and a short position worth $-x$ in this new portfolio. This strategy is self-financing, and since both portfolios are well-diversified and have the same betas, the strategy is free of risk as well. It is therefore profitable for sure.

4. Stocks, Options, and Contingent Claims

At $t = 0$, a contingent claim that pays off one dollar in state $i = 1$ at $t = 1$ and zero otherwise sells for $q^1 = 0.60$. Similarly, a contingent claim for state $i = 2$ at $t = 1$ sells for $q^2 = 0.20$ at $t = 0$ and a contingent claim for state $i = 3$ at $t = 1$ sells for $q^3 = 0.15$ at $t = 0$.

a. The payoffs from a share of stock that can be sold for $P^1_s = 1$ in state $i = 1$ at $t = 1$, $P^2_s = 2$ in state $i = 2$ at $t = 1$, and $P^3_s = 3$ in state $i = 3$ at $t = 1$ can be replicated by a portfolio that consists of $P^i_s$ contingent claims for each state $i = 1, 2, 3$. The price $P^0_s$ at which the share of stock should trade at $t = 0$ can therefore be found by calculating the cost of this portfolio of contingent claims:

$$P^0_s = P^1_s q^1 + P^2_s q^2 + P^3_s q^3 = 1 \times 0.60 + 2 \times 0.20 + 3 \times 0.15 = 1.45.$$

b. A call option that gives the owner the right, but not the obligation, to purchase one share of stock at $t = 1$ at the strike price $K = 2$ is worth $C^1 = 0$ in state $i = 1$ at $t = 1$, $C^2 = 0$ in state $i = 2$ at $t = 1$, and $C^3 = 1$ in state $i = 3$ at $t = 1$.

c. The payoff structure for the call described in part (b), above, is the same as the payoff structure for a contingent claim for state $i = 3$. Therefore, the price of the call option must equal the price of a contingent claim for state $i = 3$: $V_0 = q^3 = 0.15$. 


1. Portfolio Allocation and the Gains from Diversification

Consider a portfolio formed from two risky assets, the first with expected return equal to $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 2$ and the second with expected return equal to $\mu_2 = 5$ and standard deviation of its return equal to $\sigma_2 = 2$. Let $w$ denote the fraction of wealth in the portfolio allocated to asset 1 and $1-w$ the corresponding fraction of wealth allocated to asset 2.

a. Calculate the expected return and the standard deviation of the return on the portfolio that sets $w = 1/2$, assuming that the correlation between the two returns is $\rho_{12} = 0$.

b. Now suppose there is a third asset, with expected return equal to $\mu_3 = 7.5$ and standard deviation of its return equal to $\sigma_3 = 2$. Assume that the return on this third asset is uncorrelated with the returns on the first two assets, so that $\rho_{13} = 0$ and $\rho_{23} = 0$. Calculate the expected return and the standard deviation of the return on the portfolio that allocates equal, one-third shares of wealth to each of the three assets.

c. Suppose, finally, that there is a fourth asset, with expected return equal to $\mu_4 = 7.5$ and standard deviation of its return equal to $\sigma_4 = 2$. Assume that the return on this fourth asset is uncorrelated with the returns on the other three assets, so that $\rho_{14} = 0$, $\rho_{24} = 0$, and $\rho_{34} = 0$. Calculate the expected return and the standard deviation of the return on the portfolio that allocates equal, one-fourth shares of wealth to each of the four assets.
2. The Capital Asset Pricing Model

Suppose that the random return $\tilde{r}_M$ on the market portfolio has expected value $E(\tilde{r}_M) = 0.07$ and variance $\sigma^2_M = 0.02$ and that the return on risk-free assets is $r_f = 0.02$.

a. According to the capital asset pricing model, if a risky asset has a random return $\tilde{r}_j$ with expected value $E(\tilde{r}_j) = 0.145$, what is the numerical value of this asset’s beta $\beta_j$?

b. According to the capital asset pricing model, what is the numerical value of the covariance between this risky asset’s random return $\tilde{r}_j$ and the random return $\tilde{r}_M$ on the market portfolio?

c. According to the capital asset pricing model, if another risky asset has a random return $\tilde{r}_k$ with expected value $E(\tilde{r}_k) = 0.01$, what is the numerical value of this asset’s beta $\beta_k$?
3. A Two-Factor Arbitrage Pricing Theory

Consider an economy in which the random return $\hat{r}_i$ on each individual asset $i$ is given by

$$\hat{r}_i = E(\hat{r}_i) + \beta_{i,m}[\hat{r}_M - E(\hat{r}_M)] + \beta_{i,v}[\hat{r}_V - E(\hat{r}_V)] + \varepsilon_i$$

where, as we discussed in class, $E(\hat{r}_i)$ equals the expected return on asset $i$, $\hat{r}_M$ is the random return on the market portfolio, $\hat{r}_V$ is the random return on a “value” portfolio that takes a long position in shares of stock issued by smaller, overlooked companies or companies with high book-to-market values and a corresponding short position in shares of stock issued by larger, more popular companies or companies with low book-to-market values, $\varepsilon_i$ is an idiosyncratic, firm-specific component, and $\beta_{i,m}$ and $\beta_{i,v}$ are the “factor loadings” that measure the extent to which the return on asset $i$ is correlated with the return on the market and value portfolios. Assume, as Stephen Ross did when developing the arbitrage pricing theory (APT), that there are enough individual assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. Consider, first, a well-diversified portfolio that has $\beta_{w,m} = \beta_{w,v} = 0$. Write down the equation, implied by this version of the APT, that links the expected return $E(\hat{r}^1_w)$ on this portfolio to the return $r_f$ on a portfolio of risk-free assets. Consider, next, two more well-diversified portfolios: portfolio two with $\beta_{w,m} = 1$ and $\beta_{w,v} = 0$ and portfolio three with $\beta_{w,m} = 0$ and $\beta_{w,v} = 1$. Write down the equations, implied by this version of the APT, for the expected returns $E(\hat{r}^2_w)$ and $E(\hat{r}^3_w)$ on each of these two additional portfolios.

b. Suppose you find a fourth well-diversified portfolio that has non-zero values of both $\beta_{w,m}$ and $\beta_{w,v}$. Write down the equation, implied by this version of the APT, for the expected return $E(\hat{r}^4_w)$ on this portfolio.

c. Suppose that the expected return on the well-diversified portfolio described in part (b), above, is lower than the expected return given in your answer. Describe briefly (a sentence or two is all that it should take) the trading opportunity provided by this discrepancy that is free of risk, self-financing, but profitable for sure.
4. Stocks, Bonds, and Options Pricing

Consider a version of the no-arbitrage variant of the Arrow-Debreu pricing model in which there are two periods: $t = 0$, when investment decisions are made, and $t = 1$, when investment payoffs become known and are received. Looking ahead from period $t = 0$, there are two possible states, $i = 1$ and $i = 2$, at $t = 1$. Suppose that two complex assets are traded. One is a share of stock, which can be purchased for $P^0_s = 1.25$ at $t = 0$ and sold for $P^1_s = 3$ in state $i = 1$ at $t = 1$ and $P^2_s = 1$ in state $i = 2$ at $t = 1$. The other is a bond, which can be purchased for $Q^0 = 0.75$ at $t = 0$ and pays off one dollar for sure, that is, in both states $i = 1$ and $i = 2$ at $t = 1$.

a. Now consider a call option that gives the owner the right, but not the obligation, to purchase one share of stock at $t = 1$ at the strike price $K = 2$. Let $C^1$ and $C^2$ be the payoffs that an investor who purchases this call option at $t = 0$ receives in states $i = 1$ and $i = 2$ at $t = 1$. What are the numerical values of $C^1$ and $C^2$?

b. Suppose that an investor wants to form a portfolio at $t = 0$ consisting of $S$ shares of stock and $B$ bonds that will replicate the payoffs that he or she will receive by buying the call option described in part (a), above. What numerical values of $S$ and $B$ should he or she choose?

c. Using your answers to part (b), above, together with the information on the stock and bond prices given previously, calculate the price $V_0$ at which the call option should trade at $t = 0$. 

1. Portfolio Allocation and The Gains From Diversification

The first asset has expected return equal to $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 2$ and the second asset has expected return equal to $\mu_2 = 5$ and standard deviation of its return equal to $\sigma_2 = 2$. A portfolio has the fraction $w$ of wealth in the portfolio allocated to asset 1 and the remaining fraction $1-w$ of wealth allocated to asset 2.

a. When the correlation between the two returns is $\rho_{12} = 0$, the expected return on the portfolio that sets $w = 1/2$ is

$$(1/2)\mu_1 + (1/2)\mu_2 = 7.5$$

and the standard deviation of the return on the same portfolio is

$$\left[(1/2)^2\sigma_1^2 + (1/2)^2\sigma_2^2\right]^{1/2} = 2^{1/2} = 1.414.$$

b. The third asset has expected return equal to $\mu_3 = 7.5$ and standard deviation of its return equal to $\sigma_3 = 2$. When all of the individual returns are uncorrelated, the portfolio that allocates equal, one-third shares of wealth to each of the three assets has expected return

$$(1/3)\mu_1 + (1/3)\mu_2 + (1/3)\mu_3 = 7.5$$

and the standard deviation of the return on the same portfolio is

$$\left[(1/3)^2\sigma_1^2 + (1/3)^2\sigma_2^2 + (1/3)^2\sigma_3^2\right]^{1/2} = (4/3)^{1/2} = 1.155.$$

c. The fourth asset has expected return equal to $\mu_4 = 0.75$ and standard deviation of its return equal to $\sigma_4 = 2$. When all of the individual returns are uncorrelated, the portfolio that allocates equal, one-fourth shares of wealth to each of the four assets has expected return

$$(1/4)\mu_1 + (1/4)\mu_2 + (1/4)\mu_3 + (1/4)\mu_4 = 7.5$$

and the standard deviation of the return on the same portfolio is

$$\left[(1/4)^2\sigma_1^2 + (1/4)^2\sigma_2^2 + (1/4)^2\sigma_3^2 + (1/4)^2\sigma_4^2\right]^{1/2} = 1.$$
2. The Capital Asset Pricing Model

The random return $\tilde{r}_M$ on the market portfolio has expected value $E(\tilde{r}_M) = 0.07$ and variance $\sigma^2_M = 0.02$ and the return on risk-free assets is $r_f = 0.02$. In general, the CAPM implies that any asset with random return $\tilde{r}_j$ must have expected return

$$E(\tilde{r}_j) = r_f + \beta_j [E(\tilde{r}_M) - r_f],$$

where its beta,

$$\beta_j = \frac{\sigma_{jM}}{\sigma^2_M},$$

depends on the covariance $\sigma_{jM}$ between $\tilde{r}_j$ and $\tilde{r}_M$.

a. If a risky asset has a random return $\tilde{r}_j$ with expected value $E(\tilde{r}_j) = 0.145$, its beta must satisfy

$$0.145 = 0.02 + \beta_j(0.07 - 0.02).$$

From this equation, we can tell that $\beta_j = 0.125/0.05 = 2.5$.

b. With $\beta_j = 2.5$ and $\sigma^2_M = 0.02$, it must be that $\sigma_{jM} = 2.5 \times 0.02 = 0.05$.

c. If another risky asset has a random return $\tilde{r}_k$ with expected value $E(\tilde{r}_k) = 0.01$, its beta must satisfy

$$0.01 = 0.02 + \beta_k(0.07 - 0.02).$$

From this equation, we can tell that $\beta_k = -0.01/0.05 = -0.2$.

3. A Two-Factor Arbitrage Pricing Theory

In this version of the APT, the random return $\tilde{r}_i$ on each individual asset $i$ is given by

$$\tilde{r}_i = E(\tilde{r}_i) + \beta_{i,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{i,v}[\tilde{r}_V - E(\tilde{r}_V)] + \varepsilon_i.$$

a. The APT implies that the first well-diversified portfolio, with $\beta_{w,m} = \beta_{w,v} = 0$, will have an expected return equal to the risk-free rate,

$$E(\tilde{r}^1_w) = r_f.$$

That the second well-diversified portfolio, with $\beta_{w,m} = 1$ and $\beta_{w,v} = 0$, will have expected return equal to the expected return of the market portfolio,

$$E(\tilde{r}^2_w) = r_f + [E(\tilde{r}_M) - r_f] = E(\tilde{r}_M),$$

and that the third well-diversified portfolio, with $\beta_{w,m} = 0$ and $\beta_{w,v} = 1$, will have expected return equal to the expected return of the value portfolio,

$$E(\tilde{r}^3_w) = r_f + [E(\tilde{r}_V) - r_f] = E(\tilde{r}_V).$$
b. According to the APT, a fourth well-diversified portfolio that has non-zero values of both $\beta_{w,m}$ and $\beta_{w,v}$ will have expected return

$$E(\tilde{r}_w^4) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f].$$

c. If the expected return on portfolio four is lower than the expected return given by the solution to part (b), above, then one should form a fifth well-diversified portfolio that allocates the share $\beta_{w,m}$ of total wealth to portfolio two (or the market portfolio), the share $\beta_{w,v}$ of total wealth to portfolio three (or the value portfolio), and the remaining share $1 - \beta_{w,m} - \beta_{w,v}$ to portfolio one (or to risk-free assets). Taking a long position worth $x$ in this fifth portfolio, while simultaneously taking a short position worth $-x$ in the fourth portfolio is a trading strategy that is free of risk, self-financing, but profitable for sure.

4. Stocks, Bonds, and Options Pricing

There are two periods $t = 0$ and $t = 1$, and two possible states, $i = 1$ and $i = 2$, at $t = 1$. Two complex assets are traded. One is a share of stock, which can be purchased for $P_s^0 = 1.25$ at $t = 0$ and sold for $P_s^1 = 3$ in state $i = 1$ at $t = 1$ and $P_s^2 = 1$ in state $i = 2$ at $t = 1$. The other is a bond, which can be purchased for $Q_s^0 = 0.75$ at $t = 0$ and pays off one dollar for sure, that is, in both states $i = 1$ and $i = 2$ at $t = 1$.

a. A call option that gives the owner the right, but not the obligation, to purchase one share of stock at $t = 1$ at the strike price $K = 2$ has payoff $C^1 = 1$ in state $i = 1$ and $C^2 = 0$ in states $i = 2$ at $t = 1$.

b. If an investor wants to form a portfolio at $t = 0$ consisting of $S$ shares of stock and $B$ bonds that will replicate the payoffs that he or she will receive by buying the call option described in part (a), above, he or she should choose $S$ and $B$ to satisfy

$$3S + B = 1$$

to replicate the call’s payoff in state $i = 1$ and

$$S + B = 0$$

to replicate the call’s payoff in state $i = 2$. The second equation implies that $B = -S$. Substituting this result into the first equation reveals that

$$3S - S = 1$$

or $S = 1/2$ and $B = -1/2$.

c. Since the stock trades for $P_s^0 = 1.25$ and the bond for $Q_s^0 = 0.75$, the portfolio described in part (b), above, costs

$$(1/2) \times 1.25 - (1/2) \times 0.75 = 0.25$$

at $t = 0$. And since the call provides the same payoffs as this portfolio, it must also cost $V_0 = 0.25$ at $t = 0$. 

3
1. Stocks and Contingent Claims

Consider an economic environment with risk in which there are two periods, \( t = 0 \) and \( t = 1 \), and two possible states at \( t = 1 \): a “good” state that occurs with probability \( \pi = 1/2 \) and a “bad” state that occurs with probability \( 1 - \pi = 1/2 \). Suppose that two stocks trade in this economy. Each share in company 1 sells for \( q^1 = 5 \) at \( t = 0 \), and pays a large dividend \( d^G_1 = 3 \) in the good state at \( t = 1 \) and a small dividend \( d^B_1 = 1 \) in the bad state at \( t = 1 \). Each share in company 2 sells for \( q^2 = 8 \) at \( t = 0 \), and pays a large dividend \( d^G_2 = 4 \) in the good state at \( t = 1 \) and a small dividend \( d^B_2 = 2 \) in the bad state at \( t = 1 \). Suppose, as we have in class, that investors can take long or short positions in both assets, and there are no brokerage fees or trading costs.

a. Find the combination of purchases and/or short sales of shares in companies 1 and 2 that will replicate the payoffs on a contingent claim for the good state at \( t = 1 \).

b. Find the combination of purchases and/or short sales of shares in companies 1 and 2 that will replicate the payoffs on a contingent claim for the bad state at \( t = 1 \).

c. Find the prices at which each of the two contingent claims should sell at \( t = 0 \).

2. Pareto Optimal and Equilibrium Allocations

The graph on the next page shows an Edgeworth box, describing resource allocations in an economy with two consumers, “consumer one” and “consumer two,” and two goods, “good a” and “good b.” In the diagram, the origin for consumer one is at the bottom left, so that \( c^1_a \), consumer one’s consumption of good a, increases moving to the right along the horizontal axis, and \( c^1_b \), consumer one’s consumption of good b, increases moving up along the vertical axis. The origin for consumer two is at the top right, so that \( c^2_a \), consumer two’s consumption of good a, increases moving to the left along the horizontal axis, and \( c^2_b \), consumer two’s consumption of good b, increases moving down along the vertical axis. Both consumers prefer more to less and have a preference for variety. Thus, consumer one’s
indifference curve, traced out by the solid line, slopes down and is bowed towards the bottom left, and consumer two’s indifference curve, traced out by the dashed line, also slopes down but is bowed towards the top right.

a. Could the resource allocation shown as point A in the graph describe a Pareto optimal allocation? Explain briefly (a sentence or two is all it should take) why or why not.

b. Could the resource allocation shown as point A in the graph describe a competitive equilibrium allocation? Explain briefly (again, a sentence or two is all it should take) why or why not.

c. Starting from the resource allocation shown as point A in the graph, consider two possible reallocations:

   Reallocation 1: Take some of good a away from consumer one and give it to consumer two; and take some of good b away from consumer two and give it to consumer one.

   Reallocation 2: Take some of good b away from consumer one and give it to consumer two; and take some of good a away from consumer two and give it to consumer one.

Which of these resource reallocations would make both consumers better off: 1, 2, neither, or both? Here, you don’t have to explain; just write down an answer.
3. Pricing Risk-Free Assets

Assume, for all of the examples in this question, that there is no uncertainty: all specified payments made by all assets will be received for sure. Suppose initially that two government bonds are traded. The first is a one-year discount bond, with face (or par) value equal to $1000, which sells for $900 today, and the other is a two-year discount bond, also with face (or par) value equal to $1000, which sells for $850 today.

a. Calculate the interest rate on the one-year discount bond.

b. Calculate the annualized interest rate on the two-year bond.

c. Now suppose that the government decides to sell a third type of bond: a two-year coupon bond that makes an annual interest payment of $100 each year for the next two years then returns face value $1000 at the end of the second year. Given the prices at which the two discount bond are already trading, calculate the price at which this new, coupon bond should be expected to sell for today.

4. Criteria for Choice Over Risky Prospects

Consider once again an economic environment with risk in which there are two periods, \( t = 0 \) and \( t = 1 \), and two possible states at \( t = 1 \): a “good” state that occurs with probability \( \pi = 1/2 \) and a “bad” state that occurs with probability \( 1 - \pi = 1/2 \). In this economy, “asset one” provides a return of 3 percent in the good state and 1 percent in the bad state.

a. Calculate the expected return and the standard deviation of the return on asset one.

b. Provide an example, by writing down the expected return and standard deviation of the return, on another asset, call it “asset two,” that exhibits mean-variance dominance over asset one.

c. Provide an example, by writing down the returns provided in the good state and in the bad state, of a third asset, “asset three,” that exhibits state-by-state dominance over asset one.
5. Measuring Risk Aversion

Consider a risk-averse investor with von Neumann-Morgenstern expected utility who has initial income \( Y \) and is offered a bet: win \( kY \) with probability \( \pi \) and lose \( kY \) with probability \( 1 - \pi \). In class, we derived the approximation

\[
\pi^* \approx \frac{1}{2} + \frac{1}{4} k R_R(Y)
\]

for the probability of winning \( \pi^* \) that makes the investor indifferent between accepting and rejecting the bet, where \( R_R(Y) \) is his or her coefficient of relative risk aversion and \( k \) is the size of the bet measured as a fraction of income. Use this approximation to answer the following questions.

a. Suppose, in particular, that the consumer’s initial income is \( Y = 100000 \) and that he or she is indifferent between the bet: win 2000 with probability \( \pi^* = 0.55 \) and lose 2000 with probability \( 1 - \pi^* = 0.45 \). What is the numerical value of the consumer’s coefficient of relative risk aversion implied by the approximation?

b. Suppose that the consumer described in part (a), above, has preferences over risky alternatives that are described by a von Neumann-Morgenstern expected utility function with Bernoulli utility function

\[
u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}\]

over monetary values \( Y \) received in any particular state of the world. Given your answer to part (a), above, what value would you assign to the parameter \( \gamma \) that appears in this Bernoulli utility function?

c. Suppose that this same consumer was offered a different bet: win 4000 with probability \( \pi = 0.55 \) and lose 4000 with probability \( 1 - \pi = 0.45 \). Given his or her coefficient of relative risk aversion, which you calculated in part (a), above, would the investor accept or reject this bet?
1. Stocks and Contingent Claims

a. Shares in company 1 pay a large dividend $d^G_1 = 3$ in the good state at $t = 1$ and a small dividend $d^B_1 = 1$ in the bad state at $t = 1$. Shares in company 2 pays a large dividend $d^G_2 = 4$ in the good state at $t = 1$ and a small dividend $d^B_2 = 2$ in the bad state at $t = 1$. A contingent claim for the good state pays 1 in the good state at $t = 1$ and 0 in the bad state at $t = 1$. Thus, to replicate the payoffs on the contingent claim for the good state, an investor must purchase $s_1$ shares in company 1 and $s_2$ shares in company 2, where these two unknown values satisfy the two equations

\[ 1 = d^G_1 s_1 + d^G_2 s_2 = 3s_1 + 4s_2 \]

and

\[ 0 = d^B_1 s_1 + d^B_2 s_2 = s_1 + 2s_2. \]

Although there are many ways to solve this system of equations, perhaps the easiest is to observe that the second equation requires that

\[ s_1 = -2s_2. \]

Substituting this result into the first equation yields

\[ 1 = 3(-2s_2) + 4s_2 = -2s_2. \]

These last two equations imply

\[ s_1 = 1 \text{ and } s_2 = -1/2, \]

indicating that by buying one share in company 1 and selling short one-half share in company 2, an investor can replicate the payoffs on a contingent claim for the good state.

b. A contingent claim for the bad state pays 0 in the good state at $t = 1$ and 1 in the bad state at $t = 1$. Thus, to replicate the payoffs on the contingent claim for the bad state, an investor must purchase $s_1$ shares in company 1 and $s_2$ shares in company 2, where

\[ 0 = d^G_1 s_1 + d^G_2 s_2 = 3s_1 + 4s_2 \]

and

\[ 1 = d^B_1 s_1 + d^B_2 s_2 = s_1 + 2s_2. \]
Once again, there are many ways to solve this system of equations, but perhaps the easiest is to observe that the second equation requires that

\[ s_1 = 1 - 2s_2. \]

Substituting this result into the first equation yields

\[ 0 = 3(1 - 2s_2) + 4s_2 = 3 - 2s_2. \]

These last two equations imply

\[ s_1 = -2 \text{ and } s_2 = 3/2, \]

indicating that by selling short two shares in company 1 and buying one and one-half shares in company 2, an investor can replicate the payoffs on a contingent claim for the bad state.

c. Shares in company 1 sell for \( q^1 = 5 \) at \( t = 0 \) and shares in company 2 sell for \( q^2 = 8 \) at \( t = 0 \). The solution to part (a), above, indicates that to replicate a contingent claim for the good state, an investor needs to purchase 1 share in company 1 and sell short one-half share in company 2. Computing the cost of assembling this portfolio provides the price \( q^G \) at which a contingent claim for the good state should sell at \( t = 0 \):

\[ q^G = q^1 - (1/2)q^2 = 5 - (1/2)8 = 1. \]

The solution to part (b), above, indicates that to replicate a contingent claim for the bad state, an investor needs to purchase one and one-half shares in company 2 and sell short 2 shares in company 1. Computing the cost of assembling this portfolio provides the price \( q^B \) at which a contingent claim for the bad state should sell at \( t = 0 \):

\[ q^B = (3/2)q^2 - 2q^1 = (3/2)8 - 2(5) = 2. \]

2. Pareto Optimal and Equilibrium Allocations

a. Any Pareto optimal allocation will equate the two consumers’ marginal rates of substitution between the two goods, implying that their indifference curves in the Edgeworth box should be tangent. Since the two consumers’ indifference curves are not tangent at point A, this resource allocation cannot be Pareto optimal.

b. In any competitive equilibrium, each consumer will equate his or her marginal rate of substitution between the two goods to the relative prices of those same two goods. And since both consumers face the same relative prices, an equilibrium allocation must also equate the two consumers’ marginal rates of substitution. Since the two consumers’ indifference curves are not tangent at point A, this resource allocation cannot be part of a competitive equilibrium either.
c. Compared to the initial allocation described by point A, both consumers are better off with any other allocation that lies inside the lens-shaped region between the two indifference curves shown in the graph. Since reallocation resources in this way means taking some of good a away from consumer one and giving it to consumer two and taking some of good b away from consumer two and giving it to consumer one, reallocation 1 will make both consumers better off but reallocation 2 will not.

3. Pricing Risk-Free Assets

a. The one-year discount bond has face value equal to $1000 and sells for $900 today. The interest rate on this bond is therefore

\[ r_1 = \frac{1000}{900} - 1 = 1.1111 - 1 = 0.1111 \]

or 11.11 percent.

b. The two-year discount bond has face value equal to $1000 and sells for $850 today. The annualized interest rate on this bond is therefore

\[ r_2 = \left( \frac{1000}{850} \right)^{1/2} - 1 = 1.0847 - 1 = 0.0847 \]

or 8.47 percent.

c. The cash flows from a two-year coupon bond that makes an annual interest payment of $100 each year for the next two years then returns face value $1000 at the end of the second year can be replicated by buying 0.1 one-year discount bonds and 1.1 two-year discount bonds. Since the one-year discount bond sells for $950 today and the two-year discount bond sells for $850 today, the price \( P_C \) at which the coupon bond should sell for today is

\[ P_C = 0.1(900) + 1.1(850) = 90 + 935 = 1025. \]

4. Criteria for Choice Over Risky Prospects

a. Asset one provides a return of 3 percent in the good state and 1 percent in the bad state, each of which occurs with probability \( 1/2 \). Hence, the expected return on asset one is

\[ E(r_1) = (1/2)3 + (1/2)(1) = 3/2 + 1/2 = 2 \]

and the standard deviation of the return on asset one is

\[ \sigma(r_1) = \sqrt{[(1/2)(3 - 2)^2 + (1/2)(1 - 2)^2]} = 1. \]

b. If asset two exhibits mean-variance dominance over asset one it must have an expected return \( E(r_2) \) that is at least as large as \( E(r_1) = 2 \) and a return with standard deviation \( \sigma(r_2) \) that is no larger than \( \sigma(r_1) = 1 \). Many combinations will work, but one example is \( E(r_2) = 3 \) and \( \sigma(r_2) = 1 \).
c. If asset three exhibits state-by-state dominance over asset one, its return $r_3^G$ in the good state must be at least as large as $r_1^G = 3$ and its return $r_3^B$ in the bad state must be at least as large as $r_1^B = 1$. Once again, many combinations will work, but one example is $r_3^G = 4$ and $r_3^B = 1$.

5. Measuring Risk Aversion

a. With $Y = 100000$, the bet of 2000 represents two percent of the investor’s initial income. Hence, with $\pi^* = 0.55$ and $k = 0.02$, the approximation

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}kR_R(Y)$$

implies more specifically that

$$0.55 \approx \frac{1}{2} + (0.005)R_R(Y).$$

Hence, the investor’s coefficient of relative risk aversion $R_R(Y)$ must be

$$R_R(Y) = \frac{0.55 - 0.50}{0.005} = 10.$$

b. The Bernoulli utility function

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

implies a coefficient of relative risk aversion that is constant (independent of income) and given by $R_R(Y) = \gamma$. Hence, the answer to part (a), above, implies that $\gamma = 10$.

c. With $Y = 100000$, the new bet of 4000 represents four percent of the investor’s initial income. Hence, with $k = 0.04$ and $R_R(Y) = 10$, the approximation

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}kR_R(Y)$$

implies more specifically that

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}(0.04)(10) = 0.50 + 0.10 = 0.60$$

is the probability of winning that makes the investor indifferent between accepting and rejecting the bet. Since the probability of winning that is actually offered is $\pi = 0.55$, the investor will reject this new bet.
1. Risk Aversion and Portfolio Allocation

Consider the portfolio allocation problem faced by an investor who has initial wealth $Y_0 = 100$. The investor allocates the amount $a$ to stocks, which provide return $r_G = 0.35$ (a 35 percent gain) in a good state that occurs with probability $1/2$ and return $r_B = -0.15$ (a 15 percent loss) in a bad state that occurs with probability $1/2$. The investor allocates the remaining $Y_0 - a$ to a risk-free bond which provides the return $r_f = 0.05$ (a five percent gain) in both states. The investor has von Neumann-Morgenstern expected utility, with Bernoulli utility function of the logarithmic form

$$u(Y) = \ln(Y).$$

a. Write down a mathematical statement of this portfolio allocation problem.

b. Write down the numerical value of the investor’s optimal choice $a^*$.

c. Suppose that a second investor also has vN-M expected utility with Bernoulli utility function of the logarithmic form $u(Y) = \ln(Y)$, but has initial wealth $Y_0 = 1000$ that is ten times as large as the investor considered in parts (a) and (b) above. Still assuming that stocks provide return $r_G = 0.35$ in a good state that occurs with probability $1/2$ and return $r_B = -0.15$ in a bad state that occurs with probability $1/2$ and that the risk-free bond provides the return $r_f = 0.05$ in both states, what is the numerical value of $a^*$ measuring the amount that this second investor optimally allocates to stocks?
2. Portfolio Allocation and the Gains from Diversification

Consider portfolios formed from two risky assets, the first with expected return equal to \( \mu_1 = 10 \) and standard deviation of its return equal to \( \sigma_1 = 4 \) and the second with expected return equal to \( \mu_2 = 7 \) and standard deviation of its return equal to \( \sigma_2 = 2 \). Let \( w_1 \) denote the fraction of wealth in the portfolio allocated to asset 1 and \( w_2 \) the fraction of wealth allocated to asset 2.

a. Calculate the expected return and the standard deviation of the return on the portfolio that sets \( w_1 = 1/2 \) and \( w_2 = 1/2 \), assuming that the correlation between the two returns is \( \rho_{12} = 1 \).

b. Calculate the expected return and the standard deviation of the return on the same portfolio that sets \( w_1 = 1/2 \) and \( w_2 = 1/2 \), assuming instead that the correlation between the two returns is \( \rho_{12} = 0 \).

c. Finally, suppose that in addition to the two risky assets described above, there is also a risk-free asset with return \( r_f = 5 \). Still assuming, as in part (b), that the correlation between the two random returns is \( \rho_{12} = 0 \), calculate the expected return and the standard deviation of the return on the portfolio that allocates the fraction \( w_1 = 1/4 \) of wealth to asset 1, \( w_2 = 1/4 \) of wealth to asset 2, and the remaining fraction \( w_r = 1/2 \) to the risk-free asset.
3. Modern Portfolio Theory

The graph below traces out the minimum variance frontier from Modern Portfolio Theory. Each of the three portfolios shown, A, B, and C, lies on the minimum variance frontier: each one provides the minimized variance $\sigma_P^2$ for a given mean or expected return $\mu_P$. Portfolio A has a higher expected return than portfolio C, however, even though both returns have the same standard deviation. Thus, portfolios A and B lie on the efficient frontier, but portfolio C does not.

In answering each part of this question, assume, as Harry Markowitz did when he invented Modern Portfolio Theory, that all investors have mean-variance preferences, that is, utility functions that are increasing in their portfolio’s expected return and decreasing the variance or standard deviation of their portfolio’s random return. Assume, as well, that there is no risk-free asset, so that all investors must choose portfolios on or inside the minimum variance frontier.

a. Would any investor ever choose to hold portfolio C? Here, you can just say “yes” or “no;” you don’t have to explain why.

b. Suppose you observed one investor – call him or her “investor 1” – holding portfolio A and another investor – call him or her “investor 2” – holding portfolio B. Which investor is more risk averse: investor 1 or investor 2? Again, you can just say “investor 1” or “investor 2;” you don’t have to explain why.

c. Does portfolio A exhibit mean-variance dominance over portfolio B? Here, once more, you can just say “yes” or “no;” you don’t have to explain why.
4. The Capital Asset Pricing Model

Suppose that the random return $\tilde{r}_M$ on the market portfolio has expected value $E(\tilde{r}_M) = 0.07$ and the return on risk-free assets is $r_f = 0.02$.

a. According to the capital asset pricing model, what is the expected return on an asset with random return that has a “beta” equal to $\beta_j = 1$?

b. According to the capital asset pricing model, what is the expected return on an asset with random return that has a “beta” equal to $\beta_j = 0$?

c. According to the capital asset pricing model, what is the expected return on an asset with random return that has a “beta” equal to $\beta_j = -0.20$?

5. The Market Model and Arbitrage Pricing Theory

Consider a version of the arbitrage pricing theory that is built on the assumption that the random return $\tilde{r}_i$ on each individual asset $i$ is determined by the market model

$$
\tilde{r}_i = E(\tilde{r}_i) + \beta_i [\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i
$$

where, as we discussed in class, $E(\tilde{r}_i)$ is the expected return on asset $i$, $\tilde{r}_M$ is the return on the market portfolio and $E(\tilde{r}_M)$ is the expected return on the market portfolio, $\beta_i$ is the same beta for asset $i$ as in the capital asset pricing model, and $\varepsilon_i$ is an idiosyncratic, firm-specific component. Assume, as Stephen Ross did when developing the APT, that there are enough assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. Write down the equation, implied by the APT, for the random return $\tilde{r}_{1w}$ on a well-diversified portfolio with beta $\beta_{1w}$.

b. Write down the equation, implied by the APT, for the expected return $E(\tilde{r}_{1w})$ on this well-diversified portfolio with beta $\beta_{1w}$.

c. Suppose that you find another well-diversified portfolio with the same beta $\beta_{1w}$ that has an expected return $E(\tilde{r}_{2w})$ that is lower than the expected return $E(\tilde{r}_{1w})$ given in your answer to part (b), above. Describe briefly (a sentence or two is all that it should take) the trading opportunity provided by this discrepancy that is free of risk, self-financing, but profitable for sure.
1. Risk Aversion and Portfolio Allocation

An investor has initial wealth $Y_0 = 100$ and allocates the amount $a$ to stocks, which provide return $r_G = 0.35$ in a good state that occurs with probability $1/2$ and return $r_B = -0.15$ in a bad state that occurs with probability $1/2$. The investor allocates the remaining $Y_0 - a$ to a risk-free bond which provides the return $r_f = 0.05$ in both states. The investor has von Neumann-Morgenstern expected utility, with Bernoulli utility function of the logarithmic form

$$u(Y) = \ln(Y).$$

a. In general, the investor’s portfolio allocation problem can be stated mathematically as

$$\max_a E\{u[(1 + r_f)Y_0 + a(\tilde{r} - r_f)]\},$$

where $\tilde{r}$ is the random return on stocks, but under the particular assumptions about the Bernoulli utility function and stock returns made above, the problem can be written more specifically as

$$\max_a (1/2) \ln(105 + 0.30a) + (1/2) \ln(105 - 0.20a).$$

b. The first-order condition for the investor’s optimal choice $a^*$ is

$$\frac{(1/2)(0.30)}{105 + 0.30a^*} - \frac{(1/2)(0.20)}{105 - 0.20a^*} = 0.$$

This first-order condition leads to the numerical solution for $a^*$ as

$$\frac{(1/2)(0.30)}{105 + 0.30a^*} = \frac{(1/2)(0.20)}{105 - 0.20a^*} \Rightarrow (1/2)(0.30)(105 - 0.20a^*) = (1/2)(0.20)(105 + 0.30a^*)$$

$$0.30(105) - 0.30(0.20)a^* = 0.20(105) + 0.20 \times 0.30a^*$$

$$0.10(105) = 2(0.30)(0.20)a^*$$

$$a^* = \frac{(0.10)(105)}{2(0.30)(0.20)} = 87.5.$$
c. A second investor also has vNM expected utility with Bernoulli utility function of the logarithmic form $u(Y) = \ln(Y)$, but has initial wealth $Y_0 = 1000$ that is ten times as large as the investor considered in parts (a) and (b) above. Although it is possible to re-solve the entire problem after replacing the first investor’s $Y_0 = 100$ with the second investor’s $Y_0 = 1000$, we know from class that because the logarithmic utility function implies that the coefficient of relative risk aversion is constant, both of these investors will allocate the same fraction of their wealth to stocks. Since this fraction equals 0.875 for the first investor, it will equal 0.875 for the second investor as well. With $Y_0 = 1000$, this second investor will therefore choose $a^* = 875$.

2. Portfolio Allocation and the Gains from Diversification

Asset 1 has expected return equal to $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 4$; asset 2 has expected return equal to $\mu_2 = 7$ and standard deviation of its return equal to $\sigma_2 = 2$. The fraction of wealth in the portfolio allocated to asset 1 is $w_1$ and the fraction of wealth allocated to asset 2 is $w_2$.

a. Assuming that the correlation between the two returns is $\rho_{12} = 1$, the portfolio that sets $w_1 = 1/2$ and $w_2 = 1/2$ has expected return equal to

$$\mu_p = w_1\mu_1 + w_2\mu_2 = (1/2)10 + (1/2)7 = 5 + 3.5 = 8.5$$

and standard deviation of its return equal to

$$\sigma_p = \left[w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}\right]^{1/2}$$

$$= [(1/2)^24 + (1/2)^24 + 2(1/2)(1/2)(4)(2)(1)]^{1/2}$$

$$= (4 + 1 + 4)^{1/2} = \sqrt{9} = 3.$$ 

b. If instead the correlation between the two returns is $\rho_{12} = 0$, the same portfolio will have still have expected return $\mu_p = 8.5$, but the standard deviation of its return will equal

$$\sigma_p = \left[w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}\right]^{1/2}$$

$$= [(1/2)^216 + (1/2)^24 + 2(1/2)(1/2)(4)(2)(0)]^{1/2}$$

$$= (4 + 1)^{1/2} = \sqrt{5} = 2.24.$$ 

c. A risk-free asset has return $r_f = 5$. Still assuming, as in part (b), that the correlation between the two random returns is $\rho_{12} = 0$, the expected return on the portfolio that allocates the fraction $w_1 = 1/4$ of wealth to asset 1, $w_2 = 1/4$ of wealth to asset 2, and the remaining fraction $w_r = 1/2$ to the risk-free asset is

$$\mu_p = w_1\mu_1 + w_2\mu_2 + w_r r_f = (1/4)10 + (1/4)7 + (1/2)5 = 2.5 + 1.75 + 2.5 = 6.75.$$ 

There are a number of different ways of calculating the standard deviation of the return on this portfolio of assets, but perhaps the easiest is to note that since the two risky asset returns are assumed to be uncorrelated, and since the correlations between the
risk-free return and each of the risky returns are zero, all of the “cross-products” in the formula for the portfolio’s standard deviation equal zero, so that

$$\sigma_p = (w_1 \sigma_1^2 + w_2 \sigma_2^2)^{1/2} = [(1/4)^2 16 + (1/4)^2 4]^{1/2} = (1 + 1/4)^{1/2} = \sqrt{5}/2 = 1.12.$$  

3. Modern Portfolio Theory

The graph below traces out the minimum variance frontier from Modern Portfolio Theory.

![Graph of Modern Portfolio Theory](image)

a. No investor with mean-variance utility would ever choose to hold portfolio C, since portfolio A offers a higher expected return with the same standard deviation.

b. Investor 2, holding portfolio B, is more risk averse than investor 1, holding portfolio A, since investor 2 is accepting lower expected return in order to reduce the standard deviation of his or her portfolio’s random return.

c. No, portfolio A does not exhibit mean-variance dominance over portfolio B since, while it does have a higher expected return, the standard deviation of its return is higher as well.
4. The Capital Asset Pricing Model

The random return \( \tilde{r}_M \) on the market portfolio has expected value \( E(\tilde{r}_M) = 0.07 \) and the return on risk-free assets is \( r_f = 0.02 \).

a. According to the capital asset pricing model, the expected return on an asset with random return that has \( \beta_j = 1 \) is

\[
E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f] = 0.02 + 1(0.07 - 0.02) = 0.07.
\]

b. The expected return on an asset with random return that has \( \beta_j = 0 \) is

\[
E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f] = 0.02 + 0(0.07 - 0.02) = 0.02.
\]

c. The expected return on an asset with random return that has \( \beta_j = -0.20 \) is

\[
E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f] = 0.02 - (0.20)(0.07 - 0.02) = 0.01.
\]

5. The Market Model and Arbitrage Pricing Theory

This version of the arbitrage pricing theory is built on the assumption that the random return \( \tilde{r}_i \) on each individual asset \( i \) is determined by the market model

\[
\tilde{r}_i = E(\tilde{r}_i) + \beta_i[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i.
\]

a. The APT implies that a well-diversified portfolio with beta \( \beta_w \) will have random return

\[
\tilde{r}_w^1 = E(\tilde{r}_w^1) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)].
\]

b. The APT also implies that the well-diversified portfolio with beta \( \beta_w \) will have expected return

\[
E(\tilde{r}_w^1) = r_f + \beta_w[E(\tilde{r}_M) - r_f].
\]

c. If you find another well-diversified portfolio with the same beta \( \beta_w \) that has an expected return \( E(\tilde{r}_w^2) \) that is lower than the expected return given in the answer to part (b), you can take a long position worth \( x \) in the portfolio described in parts (a) and (b) and a short position worth \(-x\) in this new portfolio. This strategy is self-financing, and since both portfolios are well-diversified and have the same betas, the strategy is free of risk as well. It is therefore profitable for sure, and the larger the value of \( x \), the larger the profit you will make.
1. Risk Aversion and Portfolio Allocation

Consider the portfolio allocation problem faced by an investor who has initial wealth $Y_0 = 100$. The investor allocates the amount $a$ to stocks, which provide return $r_G = 0.40$ (a 40 percent gain) in a good state that occurs with probability $1/2$ and return $r_B = -0.20$ (a 20 percent loss) in a bad state that occurs with probability $1/2$. The investor allocates the remaining $Y_0 - a$ to a risk-free bond which provides the return $r_f = 0.05$ (a five percent gain) in both states. The investor has von Neumann-Morgenstern expected utility, with Bernoulli utility function of the logarithmic form

$$u(Y) = \ln(Y).$$

a. Write down a mathematical statement of this portfolio allocation problem.

b. Write down the numerical value of the investor’s optimal choice $a^\ast$.

c. Suppose that instead of the logarithmic function given above, the investor has Bernoulli utility function of the form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma},$$

with $\gamma = 1/2$. Would the solution for $a^\ast$ in this case be larger than, the same as, or smaller than, the solution for $a^\ast$ that you derived in part (b), above? Note: Here, you don’t need to actually solve for $a^\ast$, all you need to do is say how it compares to the solution from part (b).
2. Portfolio Allocation and the Gains from Diversification

Consider portfolios formed from two risky assets, the first with expected return equal to \( \mu_1 = 10 \) and standard deviation of its return equal to \( \sigma_1 = 2 \) and the second with expected return equal to \( \mu_2 = 6 \) and standard deviation of its return equal to \( \sigma_2 = 4 \). Let \( w \) denote the fraction of wealth in the portfolio allocated to asset 1, so that \( 1 - w \) is the corresponding fraction of wealth allocated to asset 2.

a. Calculate the expected return and the standard deviation of the return on the portfolio that sets \( w = 1/2 \), assuming that the correlation between the two returns is \( \rho_{12} = 0 \).

b. Calculate the expected return and the standard deviation of the return on the same portfolio that sets \( w = 1/2 \), assuming instead that the correlation between the two returns is \( \rho_{12} = -0.25 \).

c. Continue to assume, as in part (b), above, that the correlation between the two returns is \( \rho_{12} = -0.25 \). Which of the following three options would an investor with mean-variance utility prefer: (i) a portfolio with \( w = 1 \), consisting only of asset 1, (ii) a portfolio with \( w = 0 \), consisting only of asset 2, or (iii) the portfolio with \( w = 1/2 \), so that equal amounts are allocated to both assets?
3. Modern Portfolio Theory

The graph below traces out the minimum variance frontier from Modern Portfolio Theory, but also assumes that there is a risk-free asset that offers the return $r_f$. The graph shows indifference curves for two investors; in particular, $U_1$ describes the preferences of “investor 1” and $U_2$ describes the preferences of “investor 2.” The tangency points indicate that each investor is holding a larger portfolio consisting partly of the risk-free asset and partly of the “tangency portfolio” labelled T, which is the portfolio along the efficient frontier that has the highest Sharpe ratio.

![Graph showing Modern Portfolio Theory](image)

a. Which investor – investor 1 or investor 2 – is more risk averse? Here, you can just say “investor 1” or “investor 2;” you don’t have to explain why.

b. Which investor – investor 1 or investor 2 – has a larger fraction of his or her assets invested in the tangency portfolio? Again, you can just say “investor 1” or “investor 2;” you don’t have to explain why.

c. In the graph, portfolio A lies on the efficient frontier but below the line connecting the risk-free asset to the tangency portfolio. Assume, as Harry Markowitz did when he invented Modern Portfolio Theory, that all investors have mean-variance preferences, that is, utility functions that are increasing in their portfolio’s expected return and decreasing the variance or standard deviation of their portfolio’s random return. Should any such investor ever allocate all of his or her funds to portfolio A instead of holding some combination of the risk-free asset and the tangency portfolio? Here, you can just say “yes” or “no;” once more, you don’t have to explain why.
4. The Capital Asset Pricing Model

Suppose that the random return $\tilde{r}_M$ on the market portfolio has expected value $E(\tilde{r}_M) = 0.10$ and the return on risk-free assets is $r_f = 0.04$.

a. According to the capital asset pricing model, what is the “beta” $\beta_j$ on a stock with expected return equal to $E(\tilde{r}_j) = 0.10$?

b. According to the capital asset pricing model, what is the “beta” $\beta_j$ on a stock with expected return equal to $E(\tilde{r}_j) = 0.04$?

c. According to the capital asset pricing model, what is the “beta” $\beta_j$ on a stock with expected return equal to $E(\tilde{r}_j) = 0.01$?

5. The Market Model and Arbitrage Pricing Theory

Consider a version of the arbitrage pricing theory that is built on the assumption that the random return $\tilde{r}_i$ on each individual asset $i$ is determined by the market model

$$\tilde{r}_i = E(\tilde{r}_i) + \beta_i [\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i$$

where, as we discussed in class, $E(\tilde{r}_i)$ is the expected return on asset $i$, $\tilde{r}_M$ is the return on the market portfolio and $E(\tilde{r}_M)$ is the expected return on the market portfolio, $\beta_i$ is the same beta for asset $i$ as in the capital asset pricing model, and $\varepsilon_i$ is an idiosyncratic, firm-specific component. Assume, as Stephen Ross did when developing the APT, that there are enough assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. Write down the equation, implied by the APT, for the random return $\tilde{r}_w^1$ on a well-diversified portfolio with beta $\beta_w^1 = 1$.

b. Write down the equation, implied by the APT, for the expected return $E(\tilde{r}_w^1)$ on this well-diversified portfolio with beta $\beta_w^1 = 1$.

c. Suppose that you find another well-diversified portfolio with beta $\beta_w^2 = 1/2$ that has an expected return $E(\tilde{r}_w^2)$ that is equal to the expected return $E(\tilde{r}_M)$ on the market portfolio. Describe briefly (a sentence or two is all that it should take) the trading opportunity provided by this involving the two well-diversified portfolios and a portfolio of risk-free assets that is free of risk, self-financing, and profitable for sure. Here, as usual, you can assume that the expected return $E(\tilde{r}_M)$ on the market portfolio is above the risk-free rate $r_f$. 
1. Risk Aversion and Portfolio Allocation

An investor has initial wealth \( Y_0 = 100 \) and allocates the amount \( a \) to stocks, which provide return \( r_G = 0.40 \) in a good state that occurs with probability \( 1/2 \) and return \( r_B = -0.20 \) in a bad state that occurs with probability \( 1/2 \). The investor allocates the remaining \( Y_0 - a \) to a risk-free bond which provides the return \( r_f = 0.05 \) in both states. The investor has von Neumann-Morgenstern expected utility, with Bernoulli utility function of the logarithmic form

\[
u(Y) = \ln(Y).
\]

a. In general, the investor’s portfolio allocation problem can be stated mathematically as

\[
\max_a E\{u[(1 + r_f)Y_0 + a(\tilde{r} - r_f)]\},
\]

where \( \tilde{r} \) is the random return on stocks, but under the particular assumptions about the Bernoulli utility function and stock returns made above, the problem can be written more specifically as

\[
\max_a (1/2) \ln(105 + 0.35a) + (1/2) \ln(105 - 0.25a).
\]

b. The first-order condition for the investor’s optimal choice \( a^* \) is

\[
\frac{(1/2)(0.35)}{105 + 0.35a^*} - \frac{(1/2)(0.25)}{105 - 0.25a^*} = 0.
\]

This first-order condition leads to the numerical solution for \( a^* \) as

\[
\frac{(1/2)(0.35)}{105 + 0.35a^*} = \frac{(1/2)(0.25)}{105 - 0.25a^*}.
\]

\[
(1/2)(0.35)(105 - 0.25a^*) = (1/2)(0.25)(105 + 0.35a^*)
\]

\[
(0.35)(105) - (0.35)(0.25)a^* = (0.25)(105) + (0.25)(0.35)a^*
\]

\[
(0.10)(105) = 2(0.35)(0.25)a^*
\]

\[
a^* = \frac{(0.10)(105)}{2(0.35)(0.25)} = 60.
\]
c. If, instead of the logarithmic function given above, the investor has Bernoulli utility function of the form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma},$$

with $\gamma = 1/2$, the solution for $a^*$ would be larger than the value of 60 found in part (b), above. This is because the coefficient of relative risk aversion implied by this new utility function is $\gamma = 1/2$, whereas the coefficient of relative risk aversion implied by the logarithmic utility function is one. Since with this new utility function the investor is less risk averse, it is possible to conclude that $a^*$ will be larger than 60 even without solving the problem again. If you do solve for $a^*$ again, however, you will find that $a^* = 120$.

2. Portfolio Allocation and the Gains from Diversification

Asset 1 has expected return equal to $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 2$; asset 2 has expected return equal to $\mu_2 = 6$ and standard deviation of its return equal to $\sigma_2 = 4$. The fraction of wealth in the portfolio allocated to asset 1 is $w$ and the fraction of wealth allocated to asset 2 is $1 - w$.

a. Assuming that the correlation between the two returns is $\rho_{12} = 0$, the portfolio that sets $w = 1/2$ has expected return equal to

$$\mu_p = w\mu_1 + (1 - w)\mu_2 = (1/2)10 + (1/2)6 = 8$$

and standard deviation of its return equal to

$$\sigma_p = \sqrt{w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}}^{1/2} = \sqrt{(1/2)^24 + (1/2)^216}^{1/2} = (1 + 4)^{1/2} = \sqrt{5} = 2.24.$$

b. If instead the the correlation between the two returns is $\rho_{12} = -0.25$, the same portfolio will have still have expected return $\mu_p = 8$, but the standard deviation of its return will equal

$$\sigma_p = \sqrt{w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}}^{1/2} = \sqrt{(1/2)^24 + (1/2)^216 + 2(1/2)(1/2)(2)(4)(-0.25)}^{1/2} = (1 + 4 - 1)^{1/2} = \sqrt{4} = 2.$$

c. Continuing to assume, as in part (b), above, that the correlation between the two returns is $\rho_{12} = -0.25$, the portfolio with $w = 1$ will have expected return $\mu_1 = 10$ and standard deviation of its return equal to $\sigma_1 = 2$, the portfolio with $w = 0$ will have expected return $\mu_2 = 6$ and standard deviation of its return equal to $\sigma_2 = 4$, and the portfolio with $w = 1/2$ will have have expected return $\mu_p = 8$ and standard deviation of its return equal to $\sigma_p = 2$. Because it exhibits mean-variance dominance over both of the others, any investor with mean-variance utility will prefer the portfolio with $w = 1$. 2
3. Modern Portfolio Theory

The graph below traces out the minimum variance frontier from Modern Portfolio Theory, but also assumes that there is a risk-free asset that offers the return $r_f$. The graph shows indifference curves for two investors; in particular, $U_1$ describes the preferences of “investor 1” and $U_2$ describes the preferences of “investor 2.” The tangency points indicate that each investor is holding a larger portfolio consisting partly of the risk-free asset and partly of the “tangency portfolio” labelled T, which is the portfolio along the efficient frontier that has the highest Sharpe ratio.

**a.** Investor 1 is more risk averse, since he or she is accepting a lower expected return on his or her portfolio in order to reduce the standard deviation of its random return.

**b.** Investor 2 has a larger fraction of his or her assets invested in the tangency portfolio and therefore accepts a higher standard deviation in exchange for a higher expected return.

**c.** No investor with mean-variance utility will ever allocate all of his or her funds to portfolio A instead of holding some combination of the risk-free asset and the tangency portfolio. This is because there are portfolios consisting of the tangency portfolio and the risk free asset that exhibit mean-variance dominance over portfolio A.
4. The Capital Asset Pricing Model

When the random return $\tilde{r}_M$ on the market portfolio has expected value $E(\tilde{r}_M) = 0.10$ and the return on risk-free assets is $r_f = 0.04$, the capital asset pricing model implies that the expected return on any individual stock with random return $\tilde{r}_j$ is

$$E(\tilde{r}_j) = r_f + \beta_j [E(\tilde{r}_M) - r_f] = 0.04 + \beta_j (0.06).$$

Therefore, knowing the stock’s expected return is enough to determine the stock’s beta according to

$$\beta_j = \frac{E(\tilde{r}_j) - 0.04}{0.06}.$$

a. Using this formula, if $E(\tilde{r}_j) = 0.10$,

$$\beta_j = \frac{0.06}{0.06} = 1.$$

b. If $E(\tilde{r}_j) = 0.04$,

$$\beta_j = \frac{0}{0.06} = 0.$$

c. If $E(\tilde{r}_j) = 0.01$,

$$\beta_j = \frac{-0.03}{0.06} = -0.5.$$

5. The Market Model and Arbitrage Pricing Theory

This version of the arbitrage pricing theory that is built on the assumption that the random return $\tilde{r}_i$ on each individual asset $i$ is determined by the market model

$$\tilde{r}_i = E(\tilde{r}_i) + \beta_i [\tilde{r}_M - E(\tilde{r}_M)] + \xi_i.$$

a. The APT implies that the random return $\tilde{r}_w^1$ on a well-diversified portfolio with $\beta_w^1 = 1$ is

$$\tilde{r}_w^1 = E(\tilde{r}_w^1) + \beta_w^1 [\tilde{r}_M - E(\tilde{r}_M)] = E(\tilde{r}_w^1) + \tilde{r}_M - E(\tilde{r}_M).$$

b. The APT also implies that the well-diversified portfolio with $\beta_w^1 = 1$ will have expected return

$$E(\tilde{r}_w^1) = r_f + \beta_w^1 [E(\tilde{r}_M) - r_f] = E(\tilde{r}_M).$$

c. If you find another well-diversified portfolio with $\beta_w^2 = 1/2$ that has an expected return $E(\tilde{r}_w^2)$ that is equal to the expected return $E(\tilde{r}_M)$ on the market portfolio, then you should take a long position worth $x$ in this new portfolio. At the same time, you should take a short position worth $-x$ in a third portfolio that allocates half of its funds to portfolio 1, with $\beta_w^1 = 1$ and $E(\tilde{r}_w^1) = E(\tilde{r}_M)$ and half of its funds to risk-free assets. This third portfolio has the same beta $\beta_w^3 = 1/2$ as portfolio 2, but has an expected return of only $E(\tilde{r}_w^3) = (1/2)E(\tilde{r}_M) + (1/2)r_f$, which is less than $E(\tilde{r}_w^2) = E(\tilde{r}_M)$ so long as the market’s expected return is above the risk-free rate. In this case, the trading strategy is free of risk, self-financing, and profitable for sure.
1. Consumer Optimization

Consider a consumer who uses his or her income $Y$ to purchase $c_a$ apples at the price of $p_a$ per apple and $c_b$ bananas at the price of $p_b$ per banana, subject to the budget constraint

$$Y \geq p_a c_a + p_b c_b.$$ 

Suppose that the consumer’s preferences over apples and bananas are described by the utility function

$$\ln(c_a) + \beta \ln(c_b),$$

where $\ln(c)$ denotes the natural logarithm of $c$ and $\beta$ is a positive weight that captures how much the consumer likes bananas relative to apples.

a. Set up the Lagrangian for this consumer’s problem: choose $c_a$ and $c_b$ to maximize the utility function subject to the budget constraint. Then write down the two first-order conditions that characterize the consumer’s optimal choices $c_a^*$ and $c_b^*$ of how many apples and bananas to purchase.

b. Use your two first-order conditions from part (a), above, together with the budget constraint,

$$Y = p_a c_a^* + p_b c_b^*,$$

which will hold with equality when the consumer chooses $c_a^*$ and $c_b^*$ optimally, to derive solutions that show how $c_a^*$ and $c_b^*$ depend on income $Y$, the prices $p_a$ and $p_b$, and the parameter $\beta$ from the utility function. Note: To do this, you will probably also have to find an equation that shows how the value of the Lagrange multiplier $\lambda^*$ associated with the solution to the consumer’s problem also depends on $Y$, $p_a$, $p_b$, and $\beta$.

c. Use your solutions from from part (b), above, to answer the following question: Suppose that there are two consumers, both with the same utility function and budget constraint shown above. Consumer one, however, has a value of $\beta = 1$, while consumer two has a value of $\beta = 2$. Who buys more apples, consumer one or consumer two? Who buys more bananas?
2. Stocks, Bonds, and Contingent Claims

Consider an economic environment with risk in which there are two periods, \( t = 0 \) and \( t = 1 \), and two possible states at \( t = 1 \): a “good” state that occurs with probability \( \pi = 1/2 \) and a “bad” state that occurs with probability \( 1 - \pi = 1/2 \). Suppose that two assets trade in this economy. The first is a “stock,” which sells for \( q_s = 1.1 \) at \( t = 0 \) and pays a large dividend \( d_1^G = 2 \) in the good state at \( t = 1 \) and a small dividend \( d_1^B = 1 \) in the bad state at \( t = 1 \). The second is a “bond,” which sells for \( q_b = 0.9 \) at \( t = 0 \) and makes a payoff of one for sure, in both states, \( t = 1 \). Suppose, as we have done class, that investors can take long or short positions in both assets, and there are no brokerage fees or trading costs.

a. Find the combination of purchases and/or short sales of shares of stock and bonds that will replicate the payoffs on a contingent claim for the good state at \( t = 1 \).

b. Find the combination of purchases and/or short sales of shares of stock and bonds that will replicate the payoffs on a contingent claim for the bad state at \( t = 1 \).

c. Find the prices at which each of the two contingent claims should sell at \( t = 0 \) if there are to be no arbitrage opportunities across markets for stocks, bonds, and claims.

3. Pricing Safe Cash Flows

Consider an economy in which, initially, only two assets are traded. A one-year, risk-free discount bond sells for $90 today and pays off $100 for sure one year from now, and a two-year, risk-free discount bond sells for $80 today and pays off $100 two years from now.

a. Given the prices of the two discount bonds described above, what would be the price of a two-year, risk-free coupon bond that makes annual interest payments of $100 each year for the next two years and then returns face value $1000 at the end of the second year if such a bond were introduced and if there are to be no arbitrage opportunities in the bond market?

b. Still taking the prices of the two discount bonds as given and still assuming there are no arbitrage opportunities, would would be the price of another new risk-free asset, which pays off $100 for sure one year from now and $100 for sure two years from now?

c. Finally, suppose that yet another new risk-free asset begins trading, which sells for $240 today, and pays off $100 for sure one year from now, $100 for sure two years from now, and $100 for sure three years from now. Based on the prices and payoffs of this new asset, as well as those of the other risk-free assets described above, what would be the price of a three-year, risk-free discount bond that pays off $100 for sure three years from now? *Hint:* To answer this question, think first about how you could replicate the payoff on a three-year discount bond by buying the asset that pays $100 annually for each of the next three years and simultaneously short selling one or more of the other assets described above. If there are to be no arbitrage opportunities, the cost of assembling this portfolio will equal the price of the three-year discount bond.

2
4. Comparing Risky Alternatives

Suppose that two risky assets trade in an economic environment with two periods, $t = 0$ and $t = 1$, and two possible states at $t = 1$: “state A” that occurs with probability $\pi = 1/2$ and “state B” that occurs with probability $1 - \pi = 1/2$. Asset 1 pays an 8 percent return in state A and a 3 percent return in state B; whereas asset 2 pays a 4 percent return in state A and a 6 percent return in state B. This example differs slightly from those we considered in class, therefore, because state A is “good” for asset 1, whereas state B is “good” for asset 2.

a. Use the numbers from above to compute the expected return and the standard deviation of the return on each of the two assets.

b. Does one of the two asset display state-by-state dominance over the other? If so, which one dominates? Does one of the two assets display mean-variance dominance over the other? If so, which one dominates?

c. Suppose that a portfolio of these two assets can be formed by any investor who chooses to allocate one half of his or her funds to asset 1 and the other half of his or her funds to asset 2. It turns out that this portfolio will pay off a return of 6 percent (an average of 8 and 4) in state A and a return of 4.5 percent (an average of 3 and 6) in state B. Use these numbers to compute the expected return and standard deviation of the return on this portfolio. Does this portfolio display mean-variance dominance over either of the two individual assets? If so, over which one?

5. Insurance

Consider a consumer with income $100 who faces a 50 percent probability of suffering a loss that reduces his or her income to $50. Suppose that this consumer can buy an insurance policy for $x$ that protects him or her fully against this loss by paying him or her $50 to make up for the loss if it occurs. Finally, assume that the consumer has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the form

$$u(Y) = \frac{Y^{1-\gamma}}{1-\gamma},$$

with $\gamma = 2$.

a. Write down an expression for the consumer’s expected utility if he or she decides to buy the insurance.

b. Write down an expression for the consumer’s expected utility if he or she decides not to buy the insurance.

c. What is the maximum amount $x^*$ that the consumer’s will be willing to pay for the insurance policy?
1. Consumer Optimization

The consumer chooses \( c_a \) and \( c_b \) to maximize the utility function

\[
\ln(c_a) + \beta \ln(c_b)
\]

subject to the budget constraint

\[
Y \geq p_a c_a + p_a c_b.
\]

a. The Lagrangian for the consumer’s problem

\[
L = \ln(c_a) + \beta \ln(c_b) + \lambda (Y - p_a c_a - p_a c_b)
\]

leads to the first-order conditions

\[
\frac{1}{c_a^*} - \lambda^* p_a = 0
\]

and

\[
\frac{\beta}{c_b^*} - \lambda^* p_b = 0.
\]

b. Together with the budget constraint

\[
Y = p_a c_a^* + p_b c_b^*,
\]

the two first-order conditions form a system of three equations in the three unknowns: \( c_a^* \), \( c_b^* \), and \( \lambda^* \). There are many ways of solving this three-equation system, but perhaps the easiest is to rewrite the first-order conditions as

\[
c_a^* = \frac{1}{\lambda^* p_a}
\]

and

\[
c_b^* = \frac{\beta}{\lambda^* p_b}
\]

and substitute these expressions into the budget constraint to obtain

\[
Y = \frac{1}{\lambda^*} + \frac{\beta}{\lambda^*} = \frac{1 + \beta}{\lambda^*}.
\]
This last question leads directly to the solution for $\lambda^*$,

$$\lambda^* = \frac{1 + \beta}{Y},$$

which can then be substituted back to into the previous expressions for $c^*_a$ and $c^*_b$ to obtain

$$c^*_a = \frac{Y}{(1 + \beta)p_a}$$

and

$$c^*_b = \frac{\beta Y}{(1 + \beta)p_b},$$

which show how $c^*_a$ and $c^*_b$ depend on income $Y$, the prices $p_a$ and $p_b$, and the parameter $\beta$ from the utility function.

c. The solutions

$$c^*_a = \frac{Y}{(1 + \beta)p_a} \text{ and } c^*_b = \frac{\beta Y}{(1 + \beta)p_b}$$

from above, imply that consumer 1, with $\beta = 1$, will spend 1/2 of his or her income on apples and 1/2 of his or her income on bananas while consumer 2, with $\beta = 2$, will spend 1/3 of his or her income on apples and 2/3 of his or her income on bananas. Since both consumers are assumed to have the same income, consumer one buys more apples and consumer 2 buys more bananas.

2. Stocks, Bonds, and Contingent Claims

There are two periods, $t = 0$ and $t = 1$, and two possible states at $t = 1$: a “good” state that occurs with probability $\pi = 1/2$ and a “bad” state that occurs with probability $1 - \pi = 1/2$. Two assets trade in this economy. The first is a “stock,” which sells for $q^s = 1.1$ at $t = 0$ and pays a large dividend $d^G_1 = 2$ in the good state at $t = 1$ and a small dividend $d^B_1 = 1$ in the bad state at $t = 1$. The second is a “bond,” which sells for $q^b = 0.9$ at $t = 0$ and makes a payoff of one for sure, in both states, $t = 1$. Investors can take long or short positions in both assets, and there are no brokerage fees or trading costs.

a. A combination of $s$ shares of stock and $b$ bonds that replicates the payoffs on a contingent claim for the good state at $t = 1$ must have

$$sd^G + b = 2s + b = 1$$

so as to pay off one in the good state and

$$sd^B + b = s + b = 0$$

so as to pay off zero in the bad state. Subtracting the second of these two conditions from the first yields the solution

$$s = 1.$$
Substituting this solution for $s$ back into the second equation yields

$$b = -1.$$  

Evidently, a investor must purchase one share of stock and sell short one bond to replicate the claim for the good state.

b. A combination of $s$ shares of stock and $b$ bonds that replicates the payoffs on a contingent claim for the bad state at $t = 1$ must have

$$sd^G + b = 2s + b = 0$$

so as to pay off zero in the good state and

$$sd^B + b = s + b = 1$$

so as to pay off one in the bad state. Subtracting the second of these two conditions from the first yields the solution

$$s = -1.$$  

Substituting this solution for $s$ back into the second equation yields

$$b = 2.$$  

Evidently, a investor must purchase two bonds and sell short one share of stock to replicate the claim for the bad state.

c. The cost of assembling the portfolio with $s = 1$ and $b = -1$ that replicates the payoffs on the contingent claim for the good state is

$$sq^a + bq^b = q^a - q^b = 1.1 - 0.9 = 0.2.$$  

Hence, if there is to be no arbitrage, the claim for the good state must have price $q^G = 0.2$. Similarly, the cost of assembling the portfolio with $s = -1$ and $b = 2$ that replicates the payoffs on the contingent claim for the bad state is

$$sq^a + bq^b = -q^a + 2q^b = -1.1 - 2(0.9) = -1.1 + 1.8 = 0.7.$$  

Hence, the claim for the bad state must have price $q^B = 0.7$.

3. Pricing Safe Cash Flows

Initially, two assets are traded. A one-year, risk-free discount bond sells for $90 today and pays off $100 for sure one year from now, and a two-year, risk-free discount bond sells for $80 today and pays off $100 two years for sure from now.

a. The cash flow from a risk-free coupon bond that makes annual interest payments of $100 each year for the next two years and then returns face value $1000 at the end of the second year can be replicated by forming a portfolio consisting of one one-year discount bond and 11 two-year discount bonds. If there are no arbitrage opportunities, the price of this coupon bond must equal the cost of assembling this portfolio:

$$P^C_2 = (1)$90 + (11)$80 = $970.$$
b. The cash flows from another new risk-free asset, which pays off $100 for sure one year from now and $100 for sure two years from now, can be replicated by forming a portfolio consisting of one one-year discount bond and one two-year discount bond. If there are no arbitrage opportunities, the price of this new asset must equal the cost of assembling the portfolio:

\[ P_A = (1)90 + (1)80 = 170. \]

c. When yet another new risk-free asset begins trading, which sells for $240 today, and pays off $100 for sure one year from now, $100 for sure two years from now, and $100 for sure three years from now, the payoff on a three-year, risk-free discount bond can be replicated in two ways. The first possibility is to buy the new risk free asset and sell short the one described in part (b), above. The cost of assembling this portfolio is $240 − $170 = $70. The second possibility is to buy the new risk-free asset and sell short both of the two discount bonds that were initially trading: the one and two-year discount bonds. The cost of assembling this portfolio is $240 − $90 − $80 = $70. Either way, the answer is the same: the three-year discount bond should sell for \( P_3 = 70 \).

4. Comparing Risky Alternatives

Two risky assets trade in an economic environment with two periods, \( t = 0 \) and \( t = 1 \), and two possible states at \( t = 1 \): “state A” that occurs with probability \( \pi = 1/2 \) and “state B” that occurs with probability \( 1 - \pi = 1/2 \). Asset 1 pays an 8 percent return in state A and a 3 percent return in state B; whereas asset 2 pays a 4 percent return in state A and a 6 percent return in the state B.

a. Asset 1 has expected return that is a probability-weighted average of its returns across the two states,

\[ E(R_1) = (1/2)8 + (1/2)3 = 5.5, \]

and a standard deviation of its return that equals the square root of the probability-weighted average of the squared deviations of its returns from the expected value across the two states,

\[ \sigma(R_1) = [(1/2)(8 - 5.5)^2 + (1/2)(3 - 5.5)^2]^{1/2} = [(1/2)(2.5)^2 + (1/2)(-2.5)^2]^{1/2} = 2.5. \]

Likewise, asset 2 has expected return

\[ E(R_2) = (1/2)4 + (1/2)6 = 5 \]

and a standard deviation of its return that equals

\[ \sigma(R_2) = [(1/2)(4 - 5)^2 + (1/2)(6 - 5)^2]^{1/2} = [(1/2)(-1)^2 + (1/2)(1)^2]^{1/2} = 1. \]

b. Neither asset displays state-by-state dominance over the other, because asset 1 pays a higher return in state A while asset 2 pays a higher return in state B. Nor does either asset display mean-variance dominance over the other, because while asset 1 has a higher expected return, the standard deviation of its return is also higher.
c. A portfolio consisting of both assets in equal shares offers a return of 6 percent in state A and 4.5 percent in state B. Therefore, this portfolio has expected return

\[ E(R_p) = \frac{1}{2}6 + \frac{1}{2}4.5 = 5.25 \]

and a standard deviation of its return that equals

\[ \sigma(R_p) = \left[ \frac{1}{2}(6 - 5.25)^2 + \frac{1}{2}(4.5 - 5.25)^2 \right]^{1/2} = 0.75. \]

Interestingly, this portfolio dominates asset 2 purchased individually in the mean-variance sense, because it offers both a higher expected return and a lower standard deviation.

5. Insurance

The consumer has income $100 and faces a 50 percent probability of suffering a loss that reduces his or her income to $50. The consumer can buy an insurance policy for $x that protects him or her fully against this loss by paying him or her $50 to make up for this loss if it occurs, and the consumer has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the form

\[ u(Y) = \frac{Y^{1-\gamma}}{1-\gamma}, \]

with \( \gamma = 2. \)

a. If the consumer decides to buy the insurance, his or her expected utility is

\[ \frac{(100 - x)^{-1}}{-1} = -\frac{1}{100 - x}. \]

b. If the consumer decides not to buy the insurance, his or her expected utility is

\[ \frac{1}{2} \left( \frac{100^{-1}}{-1} \right) + \frac{1}{2} \left( \frac{50^{-1}}{-1} \right) = -(1/2) \left( \frac{1}{100} + \frac{1}{50} \right) = -\frac{3}{200}. \]

c. The maximum amount \( x^* \) that the consumer will be willing to pay for the insurance policy can be found by equating expected utility with insurance to expected utility without insurance, based on the answers to parts (a) and (b), above. This results in

\[ -\frac{1}{100 - x^*} = -\frac{3}{200}, \]

\[ 100 - x^* = \frac{200}{3}, \]

\[ x^* = 100 - \frac{200}{3} = \frac{100}{3} = 33.33. \]

Therefore, consumer will buy insurance if it costs less than $33.33 and will choose to remain uninsured if the policy costs more than $33.33.
This exam has five questions on four pages; before you begin, please check to make sure that your copy has all five questions and all four pages. The five questions will be weighted equally in determining your overall exam score.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

1. Risk Aversion, Wealth, and Portfolio Allocation

Consider two investors, investor $i = 1$ and investor $i = 2$, each of whom has to decide how to divide up his or her initial wealth $Y^i_0$ into an amount $a_i$ to be allocated to risky stocks and an amount $Y^i_0 - a_i$ to be allocated to perfectly safe government bonds instead. Let $\tilde{r}$ denote the random return on stocks, let $r_f$ denote the risk-free rate of return on government bonds, and assume that $E(\tilde{r}) - r_f > 0$, so that the expected return on stocks exceeds the risk-free rate. Suppose that each investor $i$ chooses $a_i$ to maximize a von Neumann-Morgenstern expected utility function with Bernoulli utility function of the natural logarithmic form

$$u(Y^i_1) = \ln(Y^i_1),$$

where $Y^i_1$ denotes the investor’s terminal wealth. Therefore, each investor, $i = 1$ and $i = 2$, solves the portfolio allocation problem

$$\max_{a_i} E\{\ln[(1 + r_f)Y^i_0 + a_i(\tilde{r} - r_f)]\}.$$ 

a. Suppose that investor $i = 1$ has initial wealth $Y^1_0 = 100$ and investor $i = 2$ has initial wealth $Y^2_0 = 1000$. Will $a^*_1$, the absolute dollar amount that investor $i = 1$ allocates to stocks, be larger than, smaller than, or the same as $a^*_2$, the absolute dollar amount that investor $i = 2$ allocates to stocks? Note: To answer this question, you don’t have to actually find the numerical values of $a^*_1$ and $a^*_2$, you only need to say whether $a^*_1$ is larger than, smaller than, or equal to $a^*_2$.

b. Continue to assume that investor $i = 1$ has initial wealth $Y^1_0 = 100$ and investor $i = 2$ has initial wealth $Y^2_0 = 1000$. Will $w^*_1 = a^*_1/Y^1_0$, the share of wealth that investor $i = 1$ allocates to stocks, be larger than, smaller than, or the same as $w^*_2 = a^*_2/Y^2_0$, the share of wealth that investor $i = 2$ allocates to stocks? Note: Again, to answer this question, you don’t have to actually find the numerical values of $w^*_1$ and $w^*_2$, you only need to say whether $w^*_1$ is larger than, smaller than, or equal to $w^*_2$. 

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c. Suppose now that both investors have the same amount of initial wealth, so that \( Y_0^1 = Y_0^2 = 100 \), but that instead of having logarithmic Bernoulli utility functions, investor \( i = 1 \) has Bernoulli utility function

\[
u_1(Y_1^1) = \left( \frac{Y_1^1}{2} \right) - 1,
\]

while investor \( i = 2 \) has Bernoulli utility function

\[
u_2(Y_1^2) = \left( \frac{Y_1^2}{4} \right) - 1.
\]

Note that both of these Bernoulli utility functions are of the general form

\[
u(Y_1) = \left( \frac{Y_1^{1-\gamma} - 1}{1 - \gamma} \right),
\]

but the specific values of \( \gamma \) imply that investor \( i = 1 \) has a constant coefficient of relative risk aversion equal to 3 whereas investor \( i = 2 \) has constant coefficient of relative risk aversion equal to 5. In this case, will \( a_1^* \), the absolute dollar amount that investor \( i = 1 \) allocates to stocks, be larger than, smaller than, or the same as \( a_2^* \), the absolute dollar amount that investor \( i = 2 \) allocates to stocks? Note: As in part (a), above, you don’t have to actually find the numerical values of \( a_1^* \) and \( a_2^* \), you only need to say whether \( a_1^* \) is larger than, smaller than, or equal to \( a_2^* \).

2. The Gains from Diversification

Consider an investor who is able to form portfolios of two risky assets by allocating the share \( w \) of his or her initial wealth to risky asset 1, with expected return \( \mu_1 = 5 \) and standard deviation of its risky return equal to \( \sigma_1 = 2 \), and allocating the remaining share \( 1 - w \) of his or her initial wealth to risky asset 2, with expected return \( \mu_2 = 3 \) and standard deviation of its risky return equal to \( \sigma_2 = 1 \). Suppose that the correlation between the two assets’ random returns is \( \rho_{12} = -1 \).

a. What will the expected return \( \mu_P \) on the investor’s portfolio be if he or she splits his or her funds evenly between the two risky assets, so that \( w = 1 - w = 1/2 \)?

b. What will the standard deviation \( \sigma_P \) of the risky return on the investor’s portfolio be if he or she splits his or her funds evenly between the two risky assets, so that \( w = 1 - w = 1/2 \)?

c. What value of \( w \) should the investor choose if he or she wishes to eliminate risk from his or her portfolio entirely?
Consider an investor whose preferences are described by a utility function defined directly over the mean \( \mu_P \) and variance \( \sigma_P^2 \) of the random return that he or she earns from his or her portfolio; suppose, in particular, that this utility function takes the form

\[ U(\mu_P, \sigma_P^2) = \mu_P - \left( \frac{A}{2} \right) \sigma_P^2. \]

The investor forms this portfolio by allocating the fraction \( w_1 \) of his or her initial wealth to risky asset 1, with expected return \( E(\tilde{r}_1) \) and variance of its risky return \( \sigma_1^2 \), fraction \( w_2 \) to risky asset 2, with expected return \( E(\tilde{r}_2) \) and variance of its risky return \( \sigma_2^2 \), and the remaining fraction \( 1 - w_1 - w_2 \) to risk-free assets with return \( r_f \). The mean return on the investor’s portfolio is therefore

\[ \mu_P = (1 - w_1 - w_2)r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2) \]

and, under the additional assumption that the random returns on the two risky assets are uncorrelated, the variance of the return on the investor’s portfolio is

\[ \sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2. \]

Thus, the investor solves the portfolio allocation problem

\[ \max_{w_1, w_2} (1 - w_1 - w_2)r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2) - \left( \frac{A}{2} \right) (w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2). \]

a. Write down the first-order conditions that determine the investor’s optimal choices \( w_1^* \) and \( w_2^* \) for the two portfolio shares \( w_1 \) and \( w_2 \). Then use the first-order conditions to derive expressions that show how the portfolio shares \( w_1^* \) and \( w_2^* \) depend on the expected returns \( E(\tilde{r}_1) \) and \( E(\tilde{r}_2) \) on the risky assets, the risk-free rate \( r_f \), the variances \( \sigma_1^2 \) and \( \sigma_2^2 \) of the two risky returns, and the risk aversion parameter \( A \) from the utility function.

b. Suppose, in particular, that \( E(\tilde{r}_1) = 10, \sigma_1^2 = 4, E(\tilde{r}_2) = 6, \sigma_2^2 = 8, r_f = 2, \) and \( A = 4. \)

What are the optimal choices \( w_1^*, w_2^*, \) and \( 1 - w_1^* - w_2^* \) in this case?

c. Note that with the specific numbers set in part (b), above, risky asset 1 dominates risky asset 2 by the mean-variance criterion, since it offers a higher expected return \emph{and} a lower variance of its return. Would any investor with mean-variance utility ever allocate any funds to risky asset 2 (that is, choose a value of \( w_2^* > 0 \)) when he or she can always allocate those funds to asset 1 instead? Explain briefly (a sentence or two is all that it should take) why or why not.
4. The Capital Asset Pricing Model

Suppose the expected return on the stock market as a whole is \( E(\hat{r}_M) = 8 \), the variance of the random return on the stock market as a whole is \( \sigma^2_M = 4 \), and the risk-free rate is \( r_f = 2 \). Use the Capital Asset Pricing Model (CAPM) to answer the following questions.

a. What is the value \( \beta_j \) for the CAPM beta on an individual stock with a random return \( \hat{r}_j \) that has variance \( \sigma^2_j = 1 \) and covariance \( \sigma_{jM} = 2 \) with the market’s random return \( \hat{r}_M \)?

b. What is the expected return \( E(\hat{r}_j) \) on the individual stock from part (a), above, with \( \sigma^2_j = 1 \) and \( \sigma_{jM} = 2 \).

c. Continue to assume, as in parts (a) and (b) above, that the individual stock’s random return has covariance \( \sigma_{jM} = 2 \) with the market’s random return, but suppose now that the individual stock’s random return has variance \( \sigma^2_j = 2 \) – twice as large as the variance from parts (a) and (b). What is the expected return on the individual stock equal to now?

5. The Market Model and Arbitrage Pricing Theory

Consider a version of the arbitrage pricing theory that is built on the assumption that the random return \( \hat{r}_i \) on each individual asset \( i \) is determined by the market model

\[
\hat{r}_i = E(\hat{r}_i) + \beta_i[\hat{r}_M - E(\hat{r}_M)] + \varepsilon_i
\]

where, as we discussed in class, \( E(\hat{r}_i) \) is the expected return on asset \( i \), \( \hat{r}_M \) is the return on the market portfolio and \( E(\hat{r}_M) \) is the expected return on the market portfolio, \( \beta_i \) is the same beta for asset \( i \) as in the capital asset pricing model, and \( \varepsilon_i \) is an idiosyncratic, firm-specific component. Assume, as Stephen Ross did when developing the APT, that there are enough assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. Write down the equation, implied by the APT, for the random return \( \hat{r}_w \) on a well-diversified portfolio with beta \( \beta_w \).

b. Write down the equation, implied by the APT, for the expected return \( E(\hat{r}_w) \) on this well-diversified portfolio with beta \( \beta_w \).

c. Suppose that you find another well-diversified portfolio with the same beta \( \beta_w \) that has an expected return that is higher than the expected return given in your answer to part (b), above. Describe briefly (a sentence or two is all that it should take) the trading opportunity provided by this discrepancy that is free of risk, self-financing, but profitable for sure.
1. Risk Aversion, Wealth, and Portfolio Allocation

Two investors, investor \(i = 1\) and investor \(i = 2\), solve the portfolio allocation problem

\[
\max_{a_i} E\{\ln[(1 + r_f)Y_0^i + a_i(\tilde{r} - r_f)]\},
\]

where \(Y_0^i\) denotes investor \(i\)'s initial wealth, \(a_i\) is the amount that he or she allocates to risky stocks, \(\tilde{r}\) is the random return on stocks, and \(r_f\) is the risk-free rate. Hence, both investors maximize a von-Neumann-Morgenstern expected utility function with Bernoulli utility function of the natural logarithmic form.

a. The logarithmic Bernoulli utility function implies that each investor will allocate the same fraction of his or her initial wealth to stocks. Therefore, if investor \(i = 1\) has initial wealth \(Y_0^1 = 100\) and investor \(i = 2\) has initial wealth \(Y_0^2 = 1000\), then investor \(i = 2\) will allocate ten times as many dollars to the stock market, so that \(a_1^* < a_2^*\).

b. Since, again, the logarithmic Bernoulli utility function implies that each investor will allocate the same fraction of his or her initial wealth to stocks, \(w_1^* = w_2^*\).

c. Both investors have Bernoulli utility functions of the constant relative risk aversion form, but investor \(i = 1\)'s coefficient of relative risk aversion is smaller than investor \(i = 2\)'s. Since both investors have the amount of initial wealth, this implies that investor \(i = 1\) will allocate more dollars to stocks, so that \(a_1^* > a_2^*\).

2. The Gains from Diversification

The investor allocates the share \(w\) of his or her initial wealth to risky asset \(1\), with expected return \(\mu_1 = 5\) and standard deviation of its risky return equal to \(\sigma_1 = 2\), and the remaining share \(1 - w\) to risky asset \(2\), with expected return \(\mu_2 = 3\) and standard deviation of its risky return equal to \(\sigma_2 = 1\). The correlation between the two assets’ random returns is \(\rho_{12} = -1\).

a. With \(w = 1 - w = 1/2\), the expected return on the investor’s portfolio will be

\[
\mu_P = (1/2)\mu_1 + (1/2)\mu_2 = (1/2)5 + (1/2)3 = 4.
\]

b. With \(w = 1 - w = 1/2\), the standard deviation of the return on the investor’s portfolio will be

\[
\sigma_P = \left[ (1/2)^2\sigma_1^2 + (1/2)^2\sigma_2^2 + 2(1/2)(1/2)\sigma_1\sigma_2\rho_{12} \right]^{1/2}
\]

\[
= \left[ (1/4)4 + (1/4)1 + 2(1/2)(1/2)(2)(1)(-1) \right]^{1/2}
\]

\[
= (1/4)^{1/2} = 1/2.
\]
c. To find the value of \( w \) that eliminates risk from investor’s portfolio entirely, substitute the value \( \rho_{12} = -1 \) into the general formula for the portfolio variance

\[
\sigma_P^2 = w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}
\]

to obtain

\[
\sigma_P^2 = w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 - 2w(1 - w)\sigma_1\sigma_2 = [w\sigma_1 - (1 - w)\sigma_2]^2.
\]

This last expression implies that \( \sigma_P = 0 \) when

\[
w\sigma_1 - (1 - w)\sigma_2 = 0
\]

or

\[
w = \frac{\sigma_2}{\sigma_1 + \sigma_2},
\]

and, in particular,

\[
w = \frac{1}{3}
\]

when \( \sigma_1 = 2 \) and \( \sigma_2 = 1 \).

3. Portfolio Allocation with Mean-Variance Utility

The investor solves the portfolio allocation problem

\[
\max_{w_1, w_2} (1 - w_1 - w_2)r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2) - \left(\frac{A}{2}\right) \left(w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2\right),
\]

where \( w_1 \) and \( w_2 \) are portfolio shares allocated to each of the two risky assets, \( E(\tilde{r}_1) \), \( E(\tilde{r}_2) \), \( \sigma_1^2 \), and \( \sigma_2^2 \) are the means and variances of the two risky returns, \( r_f \) is the risk-free rate, and \( A \) is a parameter describing the investor’s degree of risk aversion.

a. The first-order conditions for the optimal choices \( w_1^* \) and \( w_2^* \) are

\[
-r_f + E(\tilde{r}_1) - Aw_1^* \sigma_1^2 = 0
\]

and

\[
-r_f + E(\tilde{r}_2) - Aw_2^* \sigma_2^2 = 0,
\]

and can be re-arranged to obtain the solutions

\[
w_1^* = \frac{E(\tilde{r}_1) - r_f}{A\sigma_1^2}
\]

and

\[
w_2^* = \frac{E(\tilde{r}_2) - r_f}{A\sigma_2^2}.
\]
b. When, in particular, \( E(\tilde{r}_1) = 10, \sigma_1^2 = 4, E(\tilde{r}_2) = 6, \sigma_2^2 = 8, r_f = 2, \) and \( A = 4, \) the solutions from part (a), above, imply

\[
w_1^* = \frac{10 - 2}{4 \times 4} = \frac{8}{16} = \frac{1}{2},
\]
\[
w_2^* = \frac{6 - 2}{4 \times 8} = \frac{4}{32} = \frac{1}{8},
\]

and, therefore,

\[
1 - w_1^* - w_2^* = 1 - \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.
\]

c. Investors with mean-variance utility will sometimes allocate a share of their funds to asset 2, despite the fact that it is dominated by asset 1 according to the mean-variance criterion. In fact, in the specific example from part (b), above, the investor does choose \( w_2^* > 0. \) The reason is that, because of the gains from diversification, there are portfolios including asset 2 that are not dominated in the mean-variance sense by asset 1 alone.

4. The Capital Asset Pricing Model

The expected return on the stock market as a whole is \( E(\tilde{r}_M) = 8, \) the variance of the random return on the stock market as a whole is \( \sigma_M^2 = 4, \) and the risk-free rate is \( r_f = 2. \)

a. The value \( \beta_j \) for the CAPM beta on an individual stock with a random return \( \tilde{r}_j \) that has variance \( \sigma_j^2 = 1 \) and covariance \( \sigma_{jM} = 2 \) with the market’s random return \( \tilde{r}_M \) is

\[
\beta_j = \frac{\sigma_{jM}}{\sigma_M^2} = \frac{2}{4} = \frac{1}{2}.
\]

b. According to the CAPM, the expected return on the individual stock from part (a), above, with \( \sigma_j^2 = 1 \) and \( \sigma_{jM} = 2, \) is

\[
E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M - r_f)] = 2 + (1/2)(8 - 2) = 5.
\]

c. The CAPM beta depends on the covariance between the individual stock’s return and the return on the market as a whole, so doubling the stock return’s variance while holding the covariance fixed does not change the model’s implications. The expected return is still \( E(\tilde{r}_j) = 5. \)

5. The Market Model and Arbitrage Pricing Theory

In the version of the arbitrage pricing theory considered here, the random return \( \tilde{r}_i \) on each individual asset \( i \) is determined by the market model

\[
\tilde{r}_i = E(\tilde{r}_i) + \beta_i[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i
\]
where $E(\tilde{r}_i)$ is the expected return on asset $i$, $\tilde{r}_M$ is the return on the market portfolio and $E(\tilde{r}_M)$ is the expected return on the market portfolio, $\beta_i$ is the same beta for asset $i$ as in the capital asset pricing model, and $\epsilon_i$ is an idiosyncratic, firm-specific component. It is assumed, as well, that there are enough assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. A well-diversified portfolio contains enough individual assets to make the idiosyncratic component of its return negligibly small. Hence, according to this version of the APT, the random return $\tilde{r}_w$ on a well-diversified portfolio with beta $\beta_w$ is

$$\tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)].$$

b. The APT then implies that the expected return $E(\tilde{r}_w)$ on this well-diversified portfolio with beta $\beta_w$ must be

$$E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f].$$

c. If you find another well-diversified portfolio with the same beta $\beta_w$ but a higher expected return than the one in part (b), above, you should buy $x > 0$ dollars worth of that new portfolio and sell short $x$ dollars worth of the portfolio in part (b). Clearly, this combination of trades is self-financing, since the proceeds from the short sale covers the costs of the purchase. Moreover, since the new portfolio has random return $\tilde{r}_w^2$ with

$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)],$$

where $\Delta > 0$ is the difference between the expected return on the new portfolio and the expected return on the portfolio from part (b), the payoff on the trading strategy is given by

$$x(1 + \tilde{r}_w^2) - x(1 + \tilde{r}_w) = x\Delta > 0.$$ 

This last condition shows that the trade is free of risk and profitable for sure.
1. Consumer Optimization

Consider a consumer who uses his or her income $Y$ to purchase $c_a$ apples at the price of $p_a$ per apple, $c_b$ bananas at the price of $p_b$ per banana, and $c_o$ units of “all other goods” at the price of $p_o$ per other good, subject to the budget constraint

$$Y \geq p_a c_a + p_b c_b + p_o c_o.$$ 

Suppose that the consumer’s preferences over apples, bananas, and other goods are described by the utility function

$$\alpha \ln(c_a) + \beta \ln(c_b) + (1 - \alpha - \beta) \ln(c_o)$$

where $\ln(c)$ denotes the natural logarithm of $c$ and $\alpha$ and $\beta$ are weights, with $\alpha > 0$, $\beta > 0$, and $1 > \alpha + \beta$ that capture how much the consumer likes apples and bananas relative to all other goods.

a. Set up the Lagrangian for this consumer’s problem: choose $c_a$, $c_b$, and $c_o$ to maximize the utility function subject to the budget constraint. Then write down the three first-order conditions that characterize the consumer’s optimal choices $c_a^\ast$, $c_b^\ast$, $c_o^\ast$ of how many apples, bananas, and other goods to purchase.

b. Use your three first-order conditions from part (a), above, together with the budget constraint,

$$Y = p_a c_a^\ast + p_b c_b^\ast + p_o c_o^\ast,$$

which will hold with equality when the consumer chooses $c_a^\ast$, $c_b^\ast$, and $c_o^\ast$ optimally, to derive solutions that show how $c_a^\ast$, $c_b^\ast$, and $c_o^\ast$ depend on income $Y$, the prices $p_a$, $p_b$, and $p_o$, and the parameters $\alpha$ and $\beta$ from the utility function.

c. Use your solutions from from part (b), above, to answer the following questions: What fraction of income $Y$ does the consumer spend on apples? What fraction of income does the consumer spends on bananas?
2. Risky Assets, Safe Assets, and Contingent Claims

Consider an economic environment with risk in which there are two periods, $t = 0$ and $t = 1$, and two possible states at $t = 1$: a “good” state that occurs with probability $\pi = 1/2$ and a “bad” state that occurs with probability $1 - \pi = 1/2$. Suppose that two assets trade in this economy. The first is a risky asset: a “stock” that sells for $q_s = 2.20$ at $t = 0$ and pays a large dividend $d_G^1 = 3$ in the good state at $t = 1$ and a small dividend $d_B^1 = 2$ in the bad state at $t = 1$. The second is a safe asset: a “bond” that sells for $q_b = 0.9$ at $t = 0$ and makes a payoff of one for sure, in both states, $t = 1$. Suppose, as we have done class, that investors can take long or short positions in both assets, and there are no brokerage fees or trading costs.

a. Find the combination of purchases and/or short sales of shares of stock and bonds that will replicate the payoffs on a contingent claim for the good state at $t = 1$.

b. Find the combination of purchases and/or short sales of shares of stock and bonds that will replicate the payoffs on a contingent claim for the bad state at $t = 1$.

c. Suppose that another risky asset is traded in this economy, which makes a payment (provides a cash flow) of $C_G^1 = 2$ in the good state at $t = 1$ and a payment of $C_B^1 = 1$ in the bad state at $t = 1$. Find the price at which this risky asset should sell at $t = 0$ if there are to be no arbitrage opportunities across all markets for risky assets, safe assets, and contingent claims.

3. Pricing Safe Cash Flows

Consider an economy in which, initially, only two assets are traded. A one-year, risk-free discount bond sells for $90$ today and pays off $100$ for sure one year from now, and a two-year, risk-free discount bond sells for $80$ today and pays off $100$ two years for sure from now.

a. Given the prices of the two discount bonds described above, what would be the price today of a two-year, risk-free coupon bond that makes annual interest payments of $100$ each year for the next two years and then returns face value $1000$ at the end of the second year if such a bond were introduced and if there are no arbitrage opportunities in the bond market?

b. Still taking the prices of the two discount bonds as given and still assuming there are no arbitrage opportunities across markets for risk-free assets, what would be the price today of another new risk-free asset, which pays off $100$ for sure one year from now and $100$ for sure two years from now?

c. Finally, suppose that yet another new asset begins trading, which pays off $100$ for sure one year from now but requires the buyer to make a payment of $100$ for sure two years from now. Thus, the cash flows from this asset are $C_1 = 100$ at $t = 1$ and $C_2 = -100$ at $t = 2$. What would be the price today of this asset if there are no arbitrage opportunities across markets for risk-free assets?
4. Comparing Safe and Risky Alternatives

Suppose that an investor has preferences described by a von Neumann-Morgensten utility function $U(x, y, \pi)$ over lotteries $(x, y, \pi)$ of the form

$$U(x, y, \pi) = \pi x^{1/2} + (1 - \pi)y^{1/2}.$$ 

Therefore, the investor’s Bernoulli utility function over payments $c$ received in any particular state of the world is $u(c) = c^{1/2} = \sqrt{c}$.

a. Which of these two lotteries would this investor prefer? Lottery 1: $(2, 0, 1)$, which pays $2$ for sure. Or Lottery 2: $(4, 0, 1/2)$, which pays $4$ with probability $1/2$ and $0$ with probability $1/2$?

b. Which of these two lotteries would this investor prefer? Lottery 1: $(2, 0, 1)$, which pays $2$ for sure. Or Lottery 3: $(4, 0, 3/4)$, which pays $4$ with probability $3/4$ and $0$ with probability $1/4$?

c. Which of these two lotteries would this investor prefer? Lottery 1: $(2, 0, 1)$, which pays $2$ for sure. Or Lottery 4: $(16, 0, 1/2)$, which pays $16$ with probability $1/2$ and $0$ with probability $1/2$.

5. Insurance

Consider a consumer with income $100$ who faces a 50 percent probability of suffering a loss that reduces his or her income to $50$. Suppose that this consumer can buy an insurance policy for $x$ that protects him or her fully against this loss by paying him or her $50$ to make up for the loss if it occurs. Finally, assume that the consumer has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the form

$$u(Y) = \frac{Y^{1-\gamma}}{1-\gamma},$$

with $\gamma = 2$.

a. Write down an expression for the consumer’s expected utility if he or she decides to buy the insurance.

b. Write down an expression for the consumer’s expected utility if he or she decides not to buy the insurance.

c. What is the maximum amount $x^*$ that the consumer would be willing to pay for the insurance policy?
1. Consumer Optimization

The consumer chooses $c_a$, $c_b$, and $c_o$ to maximize the utility function

$$\alpha \ln(c_a) + \beta \ln(c_b) + (1 - \alpha - \beta) \ln(c_o)$$

subject to the budget constraint

$$Y \geq p_a c_a + p_b c_b + p_o c_o.$$

a. The Lagrangian for the consumer’s problem is

$$L = \alpha \ln(c_a) + \beta \ln(c_b) + (1 - \alpha - \beta) \ln(c_o) + \lambda (Y - p_a c_a - p_b c_b - p_o c_o).$$

The first-order conditions for the consumer’s optimal choices are

$$\frac{\alpha}{c_a^*} - \lambda^* p_a = 0,$$

$$\frac{\beta}{c_b^*} - \lambda^* p_b = 0,$$

and

$$\frac{1 - \alpha - \beta}{c_o^*} - \lambda^* p_o = 0.$$

b. Rearrange the first-order conditions so that they read

$$c_a^* = \frac{\alpha}{\lambda^* p_a},$$

$$c_b^* = \frac{\beta}{\lambda^* p_b},$$

and

$$c_o^* = \frac{1 - \alpha - \beta}{\lambda^* p_o},$$

then substitute these expressions into the budget constraint to obtain

$$Y = p_a c_a^* + p_b c_b^* + p_o c_o^* = \frac{\alpha}{\lambda^*} + \frac{\beta}{\lambda^*} + \frac{1 - \alpha - \beta}{\lambda^*} = \frac{1}{\lambda^*},$$

which implies that

$$\lambda^* = \frac{1}{Y}.$$
Finally, substitute this solution for $\lambda^*$ back into the rearranged first-order conditions to obtain

\[ c^*_a = \frac{\alpha Y}{p_a}, \]
\[ c^*_b = \frac{\beta Y}{p_b}, \]

and

\[ c^*_o = \frac{(1 - \alpha - \beta)Y}{p_o}. \]

c. The solutions from part (b), above, imply that

\[ \frac{p_a c^*_a}{Y} = \alpha \]

and

\[ \frac{p_b c^*_b}{Y} = \beta. \]

The consumer spends the fraction $\alpha$ of his or her income on apples and the fraction $\beta$ on bananas.

2. Risky Assets, Safe Assets, and Contingent Claims

A stock sells for $q^s = 2.20$ at $t = 0$ and pays a large dividend $d_{1G}^s = 3$ in the good state at $t = 1$ and a small dividend $d_{1B}^s = 2$ in the bad state at $t = 1$. A bond sells for $q^b = 0.9$ at $t = 0$ and makes a payoff of one for sure, in both states, $t = 1$.

a. To replicate a contingent claim for the good state, an investor needs to buy $s$ shares of stock and $b$ bonds to form a portfolio that pays off

\[ 3s + b = 1 \]

in the good state and

\[ 2s + b = 0 \]

in the bad state. Subtract the second equation from the first to obtain the solution

\[ s = 1, \]

then substitute this solution into either of the previous two equations to find

\[ b = -2. \]

Evidently, to replicate the payoffs on the claim for the good state, the investor needs to buy one share of stock and sell short two bonds.
b. To replicate a contingent claim for the bad state, an investor needs to buy $s$ shares of stock and $b$ bonds to form a portfolio that pays off

$$3s + b = 0$$

in the good state and

$$2s + b = 1$$

in the bad state. Subtract the second equation from the first to obtain the solution

$$s = -1,$$

then substitute this solution into either of the previous two equations to find

$$b = 3.$$  

Evidently, to replicate the payoffs on the claim for the bad state, the investor needs to sell short one share of stock and buy three bonds.

c. Suppose another risky asset is traded in this economy, which makes a payment of $C^G_t = 2$ in the good state at $t = 1$ and a payment of $C^B_t = 1$ in the bad state at $t = 1$. The easiest way to find the price at which this risk asset should sell at $t = 0$ is to observe that these cash flows can be replicated by a portfolio that can be assembled by buying one share of stock and selling short one bond. Since this portfolio costs $2.20 - 0.90$, no arbitrage requires that the risky asset sell for $1.30$ at $t = 0$.

Alternatively, one can observe that since the portfolio of the stock and bond that replicates the contingent claim for the good state costs $2.20 - 2 	imes 0.90$, a contingent claim for the good state has a price of $0.40$ implied by no arbitrage. Likewise, since the portfolio of the stock and bond that replicates the contingent claim for the bad state costs $-2.20 + 3 	imes 0.90$, a contingent claim for the bad state has a price of $0.50$ implied by no arbitrage. Since the new risky asset has cash flows that can be replicated by a portfolio consisting of two contingent claims for the good state and one contingent claim for the bad, its price at $t = 0$ implied by no arbitrage must be $2 	imes 0.40 + 0.50 = 1.30$.

3. Pricing Safe Cash Flows

Initially, only two assets are traded. A one-year, risk-free discount bond sells for $90$ today and pays off $100$ for sure one year from now, and a two-year, risk-free discount bond sells for $80$ today and pays off $100$ two years for sure from now.

a. A two-year, risk-free coupon bond that makes annual interest payments of $100$ each year for the next two years and then returns face value $1000$ at the end of the second year produces a cash flow that can be replicated by a portfolio that consists of one one-year discount bond and 11 two-year discount bonds. Thus, its price today implied by no arbitrage opportunities is

$$P^C = 1 \times 90 + 11 \times 80 = 970.$$
b. A risk-free asset that pays off $100 for sure one year from now and $100 for sure two years from now produces a cash flow that can be replicated by a portfolio that consists of one one-year discount bond and one two-year discount bond. Thus, its price today implied by no arbitrage is

\[ P^A = 1 \times 90 + 1 \times 80 = 170. \]

c. A risk-free asset that pays off $100 for sure one year from now but requires the buyer to make a payment of $100 for sure two years from now produces a cash flow that can be replicated by buying one one-year discount bond and selling short one two-year discount bond. Thus, its price today implied by no arbitrage is

\[ P^A = 1 \times 90 - 1 \times 80 = 10. \]

4. Comparing Safe and Risky Alternatives

The investor has preferences described by a von Neumann-Morgensten utility function \( U(x, y, \pi) \) over lotteries \( (x, y, \pi) \) of the form

\[ U(x, y, \pi) = \pi x^{1/2} + (1 - \pi) y^{1/2}. \]

a. Lottery 1, \((2, 0, 1)\), pays $2 for sure and provides the investor with expected utility

\[ U(2, 0, 1) = \sqrt{2} = 1.41. \]

Lottery 2, \((4, 0, 1/2)\), pays $4 with probability 1/2 and $0 with probability 1/2 and provides the investor with expected utility

\[ U(4, 0, 1/2) = (1/2)\sqrt{4} + (1/2)\sqrt{0} = 1. \]

Evidently, the investor prefers lottery 1. In fact, because these first two lotteries offer the same expected payoff, but only lottery 2 exposes the investor to risk, lottery 1 will be preferred by any risk averse investor.

b. Lottery 3, \((4, 0, 3/4)\), pays $4 with probability 3/4 and $0 with probability 1/4 and provides the investor with expected utility

\[ U(4, 0, 1/2) = (3/4)\sqrt{4} + (1/2)\sqrt{0} = 1.5. \]

Thus, the investor prefers lottery 3 to lottery 1.

c. Lottery 4, \((16, 0, 1/2)\), pays $16 with probability 1/2 and $0 with probability 1/2 and provides expected utility

\[ U(16, 0, 1/2) = (1/2)\sqrt{16} + (1/2)\sqrt{0} = 2. \]

The investor also prefers lottery 4 to lottery 1.
5. Insurance

Consider a consumer with income $100 who faces a 50 percent probability of suffering a loss that reduces his or her income to $50. Suppose that this consumer can buy an insurance policy for $x that protects him or her fully against this loss by paying him or her $50 to make up for the loss if it occurs. Finally, assume that the consumer has von Neumann-Morgenstern expected utility and a Bernoulli utility function of the form

\[ u(Y) = \frac{Y^{1-\gamma}}{1-\gamma}, \]

with \( \gamma = 2 \).

a. If the investor decides to buy the insurance, he or she has 100 \(- x\) to spend for sure, providing expected utility

\[ \frac{(100 - x)^{1-\gamma}}{1-\gamma} = -\frac{1}{100 - x}. \]

b. If the investor decides not to buy the insurance, he or she has 100 to spend with probability 1/2 but only 50 to spend with probability 1/2. Expected utility is

\[
(1/2) \left( \frac{100^{1-\gamma}}{1-\gamma} \right) + (1/2) \left( \frac{50^{1-\gamma}}{1-\gamma} \right) = \frac{1}{2} \left( -\frac{1}{100} \right) + \frac{1}{2} \left( -\frac{1}{50} \right) \\
= -\frac{3}{200}.
\]

c. The maximum amount \( x^* \) that the consumer would be willing to pay for the insurance policy can be found by equating expected utility with insurance to expected utility without insurance, based on the answers to parts (a) and (b) above:

\[ -\frac{1}{100 - x^*} = -\frac{3}{200}, \]

which implies

\[ 100 - x^* = 200/3 \]

or

\[ x^* = 100 - 200/3 = 100 - 66.67 = 33.33. \]
1. Risk Aversion and Portfolio Allocation

Consider the portfolio allocation problem faced by an investor who has initial wealth $Y_0 = 100$. The investor allocates the amount $a$ to stocks, which provide a return of $r_G = 0.14$ (14 percent) in a good state that occurs with probability 1/2 and a return of $r_B = 0.05$ (5 percent) in a bad state that occurs with probability 1/2. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides the return $r_f = 0.08$ in both states. The investor has von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the logarithmic form

$$u(Y) = \ln(Y).$$

a. Write down a mathematical statement of this portfolio allocation problem, and then write down the first-order condition for the investor’s optimal choice $a^*$. 

b. Write down the numerical value of the investor’s optimal choice $a^*$. 

c. Suppose that instead of providing the returns $r_G = 0.14$ in the good state and $r_B = 0.05$ in the bad state, stocks in this example provided the returns $r_G = 0.20$ (up 20 percent) in the good state and $r_B = -0.01$ (down 1 percent) in the bad state. Would the value of the investor’s optimal choice $a^*$ be greater than, less than, or the same as, the value of $a^*$ that you found for part (b), above? Note: To answer this part of the problem, you don’t have to actually calculate the new value of $a^*$, all you need to do is say whether it is greater than, less than, or the same as the value you obtained in part (b).
2. The Gains from Diversification

Consider an investor who forms a portfolio of two risky assets by allocating the share $w = 1/2$ of his or her initial wealth to risky asset 1, with expected return $\mu_1 = 8$ and standard deviation of its risky return equal to $\sigma_1 = 8$, and allocating the remaining share $1 - w = 1/2$ of his or her initial wealth to risky asset 2, with expected return $\mu_2 = 4$ and standard deviation of its risky return equal to $\sigma_2 = 4$.

a. What will the expected return $\mu_P$ and the standard deviation $\sigma_P$ of the return on the investor’s portfolio be if the correlation between the two asset’s returns is $\rho_{12} = 1$?

b. What will the expected return $\mu_P$ and the standard deviation $\sigma_P$ of the return on the investor’s portfolio be if the correlation between the two asset’s returns is $\rho_{12} = -1$?

c. What will the expected return $\mu_P$ and the standard deviation $\sigma_P$ of the return on the investor’s portfolio be if the correlation between the two asset’s returns is $\rho_{12} = -0.25$?
3. Portfolio Allocation with Mean-Variance Utility

Consider an investor whose preferences are described by a utility function defined directly over the mean $\mu_P$ and variance $\sigma^2_P$ of the random return that he or she earns from his or her portfolio; suppose, in particular, that this utility function takes the form

$$U(\mu_P, \sigma^2_P) = \mu_P - \left(\frac{A}{2}\right) \sigma^2_P.$$ 

The investor creates this portfolio by allocating the fraction $w_1$ of his or her initial wealth to risky asset 1, with expected return $E(\tilde{r}_1)$ and variance of its risky return $\sigma^2_1$, fraction $w_2$ to risky asset 2, with expected return $E(\tilde{r}_2)$ and variance of its risky return $\sigma^2_2$, and the remaining fraction $1 - w_1 - w_2$ to risk-free assets with return $r_f$. The mean or expected return on the investor’s portfolio is therefore

$$\mu_P = (1 - w_1 - w_2) r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2)$$

and, under the additional assumption that the correlation between the random returns on the two risky assets is $\rho_{12} = 0.5$, the variance of the return on the investor’s portfolio is

$$\sigma^2_P = w_1^2 \sigma^2_1 + w_2^2 \sigma^2_2 + w_1 w_2 \sigma_1 \sigma_2.$$ 

Thus, the investor solves the portfolio allocation problem

$$\max_{w_1, w_2} (1 - w_1 - w_2) r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2) - \left(\frac{A}{2}\right) (w_1^2 \sigma^2_1 + w_2^2 \sigma^2_2 + w_1 w_2 \sigma_1 \sigma_2).$$

a. Write down the first-order conditions that determine the investor’s optimal choices $w_1^*$ and $w_2^*$ for the two portfolio shares $w_1$ and $w_2$.

b. Suppose, in particular, that $E(\tilde{r}_1) = 6$, $\sigma^2_1 = 4$ (and therefore $\sigma_1 = 2$), $E(\tilde{r}_2) = 4$, $\sigma^2_2 = 1$ (and therefore $\sigma_2 = 1$), $r_f = 2$, and $A = 4$. What are the optimal choices $w_1^*$, $w_2^*$, and $1 - w_1^* - w_2^*$ in this case?

c. Suppose that the value of $A$ in this problem was equal to 2 instead of 4. Would you expect the investor to allocate a larger or a smaller share $1 - w_1^* - w_2^*$ of his or her initial wealth to risk-free assets? Note: To answer this part of the problem, you don’t have to actually calculate the new value of $1 - w_1^* - w_2^*$; all you need to do is to say whether it is larger or smaller than the answer you obtained in part (b), above.
4. The Capital Asset Pricing Model

Suppose that the random return $\tilde{r}_M$ on the market portfolio has expected value $E(\tilde{r}_M) = 0.07$ and variance $\sigma^2_M = 0.025$ and that the return on risk-free assets is $r_f = 0.01$.

a. According to the capital asset pricing model, what is the expected return on a risky asset with random return $\tilde{r}_j$ that has variance $\sigma^2_j = 0.25$ and a covariance $\sigma_{jM} = 0$ of zero with the random return on the market?

b. According to the capital asset pricing model, what is the expected return on the risky asset if, instead, its random return $\tilde{r}_j$ has variance $\sigma^2_j = 0.25$ and a covariance of $\sigma_{jM} = 0.025$ with the random return on the market?

c. According to the capital asset pricing model, what is the expected return on the risky asset if, instead, its random return $\tilde{r}_j$ has variance $\sigma^2_j = 0.25$ and a covariance of $\sigma_{jM} = 0.050$ with the random return on the market?

5. The Market Model and Arbitrage Pricing Theory

Consider a version of the arbitrage pricing theory that is built on the assumption that the random return $\tilde{r}_i$ on each individual asset $i$ is determined by the market model

$$\tilde{r}_i = E(\tilde{r}_i) + \beta_i [\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i$$

where, as we discussed in class, $E(\tilde{r}_i)$ is the expected return on asset $i$, $\tilde{r}_M$ is the return on the market portfolio and $E(\tilde{r}_M)$ is the expected return on the market portfolio, $\beta_i$ is the same beta for asset $i$ as in the capital asset pricing model, and $\varepsilon_i$ is an idiosyncratic, firm-specific component. Assume, as Stephen Ross did when developing the APT, that there are enough assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. Write down the equation, implied by the APT, for the random return $\tilde{r}_w$ on a well-diversified portfolio with beta $\beta_w$.

b. Write down the equation, implied by the APT, for the expected return $E(\tilde{r}_w)$ on this well-diversified portfolio with beta $\beta_w$.

c. Suppose that you find another well-diversified portfolio with the same beta $\beta_w$ that has an expected return that is higher than the expected return given in your answer to part (b), above. Describe briefly (a sentence or two is all that it should take) the trading opportunity provided by this discrepancy that is free of risk, self-financing, but profitable for sure.
1. Risk Aversion and Portfolio Allocation

An investor with initial wealth $Y_0 = 100$ allocates the amount $a$ to stocks, which provide a return of $r_G = 0.14$ (14 percent) in a good state that occurs with probability 1/2 and a return of $r_B = 0.05$ (5 percent) in a bad state that occurs with probability 1/2. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides the return $r_f = 0.08$ in both states. The investor has von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the logarithmic form

$$u(Y) = \ln(Y).$$

a. The investor’s problem can be stated mathematically as

$$\max_a 0.5 \ln[(1 + r_f)Y_0 + (r_G - r_f)a] + 0.5 \ln[(1 + r_f)Y_0 + (r_B - r_f)a]$$

or, more specifically, as

$$\max_a 0.5 \ln(108 + 0.06a) + 0.5 \ln(108 - 0.03a).$$

The first-order condition for the investor’s optimal choice $a^*$ is

$$\frac{0.5(0.06)}{108 + 0.06a^*} - \frac{0.5(0.03)}{108 - 0.03a^*} = 0.$$

b. To find numerical value of the investor’s optimal choice $a^*$, rewrite the first-order condition from part (a), above, as

$$\frac{0.5(0.06)}{108 + 0.06a^*} = \frac{0.5(0.03)}{108 - 0.03a^*}$$

and then solve in a series of steps

$$\frac{2}{108 + 0.06a^*} = \frac{1}{108 - 0.03a^*}$$

$$216 - 0.06a^* = 108 + 0.06a^*$$

$$108 = 0.12a^*$$

$$a^* = \frac{108}{0.12} = 900.$$
c. If instead of providing the returns \( r_G = 0.14 \) in the good state and \( r_B = 0.05 \) in the bad state, stocks in this example provided the returns \( r_G = 0.20 \) in the good state and \( r_B = -0.01 \) in the bad state, the investor’s optimal choice \( a^* \) will be less than 900, the value found for part (b), above. The reason is that this change in returns constitutes a mean-preserving spread: stocks become riskier, without offering a larger expected return. In response, the risk-averse investor allocates less to stocks.

2. The Gains from Diversification

An investor forms a portfolio of two risky assets by allocating the share \( w = 1/2 \) of his or her initial wealth to risky asset 1, with expected return \( \mu_1 = 8 \) and standard deviation of its risky return equal to \( \sigma_1 = 8 \), and allocating the remaining share \( 1 - w = 1/2 \) of his or her initial wealth to risky asset 2, with expected return \( \mu_2 = 4 \) and standard deviation of its risky return equal to \( \sigma_2 = 4 \). In general, the expected return on the investor’s portfolio is

\[
\mu_P = w\mu_1 + (1 - w)\mu_2 = (1/2)8 + (1/2)4 = 6,
\]

while the standard deviation of the return on the portfolio is

\[
\sigma_P = \left[ w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}\right]^{1/2}
\]

\[
= \left[ \left(\frac{1}{4}\right)64 + \left(\frac{1}{4}\right)16 + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(8)(4)\rho_{12}\right]^{1/2}
\]

\[
= \left(20 + 16\rho_{12}\right)^{1/2}.
\]

a. Thus, if the correlation between the two asset’s returns is \( \rho_{12} = 1 \), the portfolio’s expected return is \( \mu_P = 6 \) and the standard deviation of its return is \( \sigma_P = (36)^{1/2} = 6 \).

b. Likewise, if the correlation between the two asset’s returns is \( \rho_{12} = -1 \), the portfolio’s expected return is \( \mu_P = 6 \) and the standard deviation of its return is \( \sigma_P = (4)^{1/2} = 2 \).

c. Finally, if the correlation between the two asset’s returns is \( \rho_{12} = -0.25 \), the portfolio’s expected return is \( \mu_P = 6 \) and the standard deviation of its return is \( \sigma_P = (16)^{1/2} = 4 \).

3. Portfolio Allocation with Mean-Variance Utility

An investor’s preferences are described by a utility function defined directly over the mean \( \mu_P \) and variance \( \sigma_P^2 \) of the random return that he or she earns from his or her portfolio; in particular, this utility function takes the form

\[
U(\mu_P, \sigma_P^2) = \mu_P - \left(\frac{A}{2}\right)\sigma_P^2.
\]

The investor creates this portfolio by allocating the fraction \( w_1 \) of his or her initial wealth to risky asset 1, with expected return \( \tilde{r}_1 \) and variance of its risky return \( \tilde{r}_1^2 \), fraction \( w_2 \) to risky asset 2, with expected return \( \tilde{r}_2 \) and variance of its risky return \( \tilde{r}_2^2 \), and the
remaining fraction $1 - w_1 - w_2$ to risk-free assets with return $r_f$. The mean return on the investor’s portfolio is therefore

$$\mu_P = (1 - w_1 - w_2)r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2)$$

and, under the additional assumption that the correlation between the random returns on the two risky assets is $\rho_{12} = 0.5$, the variance of the return on the investor’s portfolio is

$$\sigma^2_P = w_1^2 \sigma^2_1 + w_2^2 \sigma^2_2 + w_1 w_2 \sigma_1 \sigma_2.$$

Thus, the investor solves the portfolio allocation problem

$$\max_{w_1, w_2} (1 - w_1 - w_2)r_f + w_1 E(\tilde{r}_1) + w_2 E(\tilde{r}_2) - \left(\frac{A}{2}\right) (w_1^2 \sigma^2_1 + w_2^2 \sigma^2_2 + w_1 w_2 \sigma_1 \sigma_2).$$

a. The first-order condition for the investor’s optimal choice of $w_1^*$ is

$$-r_f + E(\tilde{r}_1) - \left(\frac{A}{2}\right) (2\sigma_1^2 w_1^* + \sigma_1 \sigma_2 w_2^*) = 0,$$

and the first-order condition for the investor’s optimal choice of $w_2^*$ is

$$-r_f + E(\tilde{r}_2) - \left(\frac{A}{2}\right) (2\sigma_2^2 w_2^* + \sigma_1 \sigma_2 w_1^*) = 0.$$

b. With $E(\tilde{r}_1) = 6, \sigma_1^2 = 4, E(\tilde{r}_2) = 4, \sigma_2^2 = 1$, and $r_f = 2$, the first-order conditions from part (a), above, specialize to

$$4 = \frac{A}{2} (8w_1^* + 2w_2^*)$$

and

$$2 = \frac{A}{2} (2w_1^* + 2w_2^*).$$

Furthermore, with $A = 4$, these first-order conditions simplify further to

$$4 = 16w_1^* + 4w_2^*$$

and

$$2 = 4w_1^* + 4w_2^*.$$

These last two equations form a system of two equations in the two unknowns, $w_1^*$ and $w_2^*$, which can be solved in a variety of ways. Probably the easiest is to subtract the second from the first to obtain

$$2 = 12w_1^*$$

or

$$w_1^* = 1/6.$$

Substitute this solution back into the second equation in the system,

$$2 = 4w_1^* + 4w_2^*,$$
to get
\[ 2 = 4/6 + 4w_2^*, \]

or
\[ w_2^* = 2/6 = 1/3. \]

Thus, the investor allocates one-sixth of his or her wealth to asset 1, one-third of his or her wealth to asset 2, and the remaining fraction
\[ 1 - w_1^* - w_2^* = 1 - 1/6 - 2/6 = 3/6 = 1/2 \]
to risk-free assets.

c. Suppose that the value of \( A \) in this problem was equal to 2 instead of 4. Then the investor would be less risk averse, so one would expect the investor to allocate a smaller share \( 1 - w_1^* - w_2^* \) of his or her initial wealth to risk-free assets than in part (b), above. In fact, going back to the original system of equations
\[
4 = \frac{A}{2}(8w_1^* + 2w_2^*)
\]

and
\[
2 = \frac{A}{2}(2w_1^* + 2w_2^*),
\]

and setting \( A = 2 \) instead of \( A = 4 \) yields
\[
4 = 8w_1^* + 2w_2^*
\]

and
\[
2 = 2w_1^* + 2w_2^*.
\]

Subtracting the second equation from the first leads to the solution
\[ w_1^* = 1/3 \]

for this case, and substituting this solution for \( w_1^* \) back into the second equation yields
\[ w_2^* = 2/3. \]

Since \( w_1^* + w_2^* = 1 \), the investor with \( A = 2 \) will allocate all of his or her wealth to risky assets and none to the risk-free asset.

4. The Capital Asset Pricing Model

The random return \( \tilde{r}_M \) on the market portfolio has expected value \( E(\tilde{r}_M) = 0.07 \) and variance \( \sigma^2_M = 0.025 \) and the return on risk-free assets is \( r_f = 0.01 \). The CAPM beta for any individual asset with random return \( \tilde{r}_j \) that has a covariance \( \sigma_{jM} \) with the market is
\[
\beta_j = \frac{\sigma_{jM}}{\sigma^2_M} = \frac{\sigma_{jM}}{0.025}
\]
and, according to the CAPM, the expected return on this asset must be

\[ E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f] = 0.01 + \beta_j(0.06). \]

Taken together, these equations imply that the individual asset’s expected return does not depend at all on the variance \( \sigma_j^2 \) of that asset’s random return; instead, the expected return depends, through beta, on the covariance of the individual asset’s random return with the random return on the market.

a. If the individual asset’s random return \( \tilde{r}_j \) has covariance \( \sigma_{jM} = 0 \) of zero with the random return on the market, then the asset’s beta is zero, and its expected return according to the CAPM should be the same as the risk-free rate:

\[ E(\tilde{r}_j) = 0.01. \]

b. Likewise, if the individual asset’s random return \( \tilde{r}_j \) has covariance \( \sigma_{jM} = 0.025 \) with the random return on the market, then the asset’s beta is one, and its expected return according to the CAPM should be the same as the expected return on the market:

\[ E(\tilde{r}_j) = 0.01 + 0.06 = 0.07. \]

c. Finally, if the individual asset’s random return \( \tilde{r}_j \) has covariance \( \sigma_{jM} = 0.050 \) with the random return on the market, then the asset’s beta is two, and its expected return according to the CAPM should be

\[ E(\tilde{r}_j) = 0.01 + 2(0.06) = 0.13, \]

or 13 percent.

5. The Market Model and Arbitrage Pricing Theory

This version of the arbitrage pricing theory is built on the assumption that the random return \( \tilde{r}_i \) on each individual asset \( i \) is determined by the market model

\[ \tilde{r}_i = E(\tilde{r}_i) + \beta_i[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_i, \]

where \( E(\tilde{r}_i) \) is the expected return on asset \( i \), \( \tilde{r}_M \) is the return on the market portfolio and \( E(\tilde{r}_M) \) is the expected return on the market portfolio, \( \beta_i \) is the same beta for asset \( i \) as in the capital asset pricing model, and \( \varepsilon_i \) is an idiosyncratic, firm-specific component.

We’ll assume, as Stephen Ross did when developing the APT, that there are enough assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

a. According to the APT, a well-diversified portfolio is one in which there are enough individual assets to make idiosyncratic risk vanish. Thus, the random return \( \tilde{r}_w \) on a well-diversified portfolio with beta \( \beta_w \) is

\[ \tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]. \]
b. The APT then implies that the expected return $E(\tilde{r}_w)$ on this well-diversified portfolio with beta $\beta_w$ must be

$$E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f].$$

c. Suppose that we find another well-diversified portfolio with the same beta $\beta_w$ that has an expected return that is higher than the expected return given in your answer to part (b), above. Then according to the APT, this second well-diversified portfolio would have random return $\tilde{r}_w^2$ given by

$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)],$$

with $\Delta > 0$. By taking a long position (by buying) worth $x$ in this second portfolio and a corresponding short position (by selling short) worth $-x$ in the first, we could obtain a payoff equal to

$$x(1 + \tilde{r}_w^2) - x(1 + \tilde{r}_w) = x\Delta > 0.$$ 

This strategy is self-financing, free of risk, and profitable for sure.
1. Intertemporal Consumer Optimization

Following Irving Fisher, consider a consumer who receives income $Y_0$ in period $t = 0$ (today), which he or she divides up into an amount $c_0$ to be consumed and an amount $s$ to be saved (or borrowed, if $s < 0$), subject to the budget constraint

$$ Y_0 \geq c_0 + s. $$

Suppose that the consumer then receives income $Y_1$ in period $t = 1$ (next year), which he or she combines with his or her savings from period $t = 0$ to finance consumption $c_1$, subject to the budget constraint

$$ Y_1 + (1 + r)s \geq c_1, $$

where $r$ denotes the interest rate on both saving and borrowing. As in class, we can combine these two single-period budget constraints into one present-value budget constraint

$$ Y_0 + \frac{Y_1}{1 + r} \geq c_0 + \frac{c_1}{1 + r}, \quad (1) $$

thereby also eliminating $s$ as a separate choice variable in the consumer’s problem.

Suppose, finally, that the consumer’s preferences over consumption during the two periods are described by the utility function

$$ \ln(c_0) + \beta \ln(c_1), \quad (2) $$

where the discount factor $\beta$, satisfying $0 < \beta < 1$, measures the consumer’s patience and $\ln$ denotes the natural logarithm.

a. Set up the Lagrangian for the consumer’s problem: choose $c_0$ and $c_1$ to maximize the utility function in (2) subject to the budget constraint in (1). Then, write down the first-order conditions for $c_0$ and $c_1$ that characterize the solution to this problem.
b. Next, assume in particular that $\beta = 0.8 = 4/5$, $r = 0.25$ so that $1 + r = 1.25 = 5/4$, $Y_0 = 80$, and $Y_1 = 125$. Use these values, together with the first-order conditions you derived in answering part (a), above, and the budget constraint (1), which will hold as an equality when the consumer is choosing $c_0$ and $c_1$ optimally, to find the numerical values of $c_0^*$ and $c_1^*$ that solve the consumer’s problem.

c. Finally, continue to assume that $\beta = 0.8 = 4/5$ and $r = 0.25$ so that $1 + r = 1.25 = 5/4$, but suppose that the consumer’s income in the two periods is given instead by $Y_0 = 180$ and $Y_1 = 0$. What are the numerical values of $c_0^*$ and $c_1^*$ that solve the consumer’s problem now?

2. Implementing State-Contingent Consumption Plans

In extending Irving Fisher’s intertemporal model of consumer decision-making to incorporate risk and uncertainty, Kenneth Arrow and Gerard Debreu imagined that the consumer chooses consumption $c_0$ at $t = 0$ (today), consumption $c_G^1$ in a good state that occurs with probability $\pi$ at $t = 1$ (next year), and consumption $c_B^1$ in a bad state that occurs with probability $1 - \pi$ at $t = 1$. To implement these state-contingent consumption plans, Arrow and Debreu suggested that the consumer could trade contingent claims, taking long or short positions in these assets as needed to rearrange consumption across time periods and across future states.

Suppose, as we did in class, that a contingent claim for the good state sells for $q_G^G$ units of consumption at $t = 0$ and pays off one unit of consumption in the good state at $t = 1$ and zero in the bad state at $t = 1$. Suppose, likewise, that a contingent claim for the bad state sells for $q_B^B$ units of consumption at $t = 0$ and pays off one unit of consumption in the bad state at $t = 1$ and zero in the good state at $t = 1$.

For this problem, assume in particular that $q_G^G = 1$ and $q_B^B = 1$, so that the prices of the two contingent claims both equal one at $t = 0$. These assumptions should make it easier for you to answer the questions in parts (a)-(c), below.

a. Suppose, first, that the consumer wants to increase his or her consumption by one unit in the bad state at $t = 1$ and is willing to give up one unit of consumption at $t = 0$ in order to do so. Describe, briefly, a trading strategy involving contingent claims that the consumer can use to rearrange his or her consumption spending in this way. In describing this strategy please indicate which claim or claims – for the good state, bad state, or both – the consumer needs to trade and whether he or she should be buying or short selling that claim (or those claims) at $t = 0$.

b. Suppose, instead, that the consumer wants to increase his or her consumption by one unit at $t = 0$ and is willing to give up one unit of consumption in the good state at $t = 1$ in order to do so. What trading strategy involving contingent claims will work to rearrange the consumer’s spending in this way?
c. Suppose, finally, that the consumer wants to increase his or her consumption by one unit in the bad state at \( t = 1 \), is willing to give up one unit of consumption in the good state at \( t = 1 \) in order to do so, but wants to keep consumption at \( t = 0 \) unchanged. What trading strategy involving contingent claims will work to rearrange the consumer’s spending in this way?

3. Stocks, Bonds, and Stock Options

Consider an economy in which there are two periods \( t = 0 \) (today) and \( t = 1 \) (next year) and two states at \( t = 1 \): a good and bad state that occur with equal probabilities \( \pi = 1/2 \).

Suppose that, in this economy, two assets are traded. A risky stock sells for \( q^s = 2.10 \) at \( t = 0 \), \( P^G = 4 \) in the good state at \( t = 1 \), and \( P^B = 1 \) in the bad state at \( t = 1 \). A risk-free bond sells for \( q^b = 0.90 \) at \( t = 0 \) and pays off 1 in both the good and bad states at \( t = 1 \).

a. Now suppose that a third asset is introduced into this economy: a (call) option that sells for \( q^o \) at \( t = 0 \) and gives the holder the right, but not the obligation, to buy a share of stock at the strike price \( K = 2 \) at \( t = 1 \). What will the payoffs for this stock option be in the good and bad states at \( t = 1 \)?

b. As Robert Merton first pointed out, the payoffs on the stock option can be replicated by forming a portfolio that consists of \( s \) shares of the stock itself and \( b \) bonds, provided that traders are allowed to take both long and short positions in the stock and bond. Use your answers to part (a), above, to find the values of \( s \) and \( b \) in this case.

c. Finally, use your answers from part (b), above, to determine the price \( q^o \) of the stock option at \( t = 0 \) that will prevail if there are no arbitrage opportunities across the markets for stocks, bonds, and options.

4. Pricing Riskless Cash Flows

Consider an economy in which, initially, three risk-free discount bonds are traded. A one-year discount bond sells for \( P_1 = 0.90 \) today and pays off one dollar, for sure, one year from now. A two-year discount bond sells for \( P_2 = 0.60 \) today and pays off one dollar, for sure, two years from now. A three-year discount bond sells for \( P_3 = 0.50 \) today and pays off one dollar, for sure, three years from now.

a. Suppose, first, that a risk-free coupon bond is introduced into this economy. The coupon bond makes annual interest (coupon) payments of $10 each year, every year, for the next three years and also returns face or par value of $100 three years from now. At what price \( P^C \) will this coupon bond sell for today, if there are no arbitrage opportunities across the markets for discount and coupon bonds?
b. Suppose, next, that another risk-free asset is introduced, which makes two payments: $100 two years from now and $500 three years from now. At what price \( P^A \) will this risk-free asset sell for today, if there are no arbitrage opportunities across all markets for risk-free assets?

c. Suppose, finally, that you plan to take out a one-year loan of $1 one year from now, and want to make arrangements for this loan and lock in the interest rate today. As you saw by working through question 3 on problem set 5, you can do this by forming the appropriate portfolio of discount bonds. In particular, to obtain the $1 one year from now, you can buy a one-year discount bond for $0.90. How many two-year discount bonds will you need to sell short in order to offset the $0.90 cost of the one-year discount bonds you have to purchase? How much will you have to pay to buy back these two-year discount bonds two years from now? What is the implied interest rate (the forward rate) on the loan of $1, made one year from now and paid back two years from now?

5. Criteria for Choosing Between Risky Alternatives

Consider an economy in which two assets have percentage returns \( \tilde{R}_1 \) and \( \tilde{R}_2 \) that vary across two states that occur with equal probability as follows:

<table>
<thead>
<tr>
<th>Return on</th>
<th>State 1</th>
<th>State 2</th>
<th>( E(\tilde{R}) )</th>
<th>( \sigma(\tilde{R}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1 ( \tilde{R}_1 )</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Asset 2 ( \tilde{R}_2 )</td>
<td>9</td>
<td>3</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

To save you time and trouble, the table also reports the expected return \( E(\tilde{R}) \) and the standard deviation of the risky return \( \sigma(\tilde{R}) \) on each of the risky assets.

a. Does one asset exhibit state-by-state dominance over the other? If so, which one?

b. Does one asset exhibit mean-variance dominance over the other? If so, which one?

c. Suppose, finally, that a third asset, “Asset 3,” becomes available, with random percentage return \( \tilde{R}_3 \) that equals 10 in the good state and 4 in the bad. For this asset, \( E(\tilde{R}_3) = 7 \) and \( \sigma(\tilde{R}_3) = 3 \). If an investor who always prefers more to less has to choose one of these three assets, which one will he or she pick: asset 1, asset 2, or asset 3? Can you tell for sure, or is it ambiguous?
1. Intertemporal Consumer Optimization

The consumer chooses $c_0$ and $c_1$ to maximize the utility function

$$\ln(c_0) + \beta \ln(c_1)$$

subject to the budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$ 

a. The Lagrangian for the consumer’s problem is

$$L = \ln(c_0) + \beta \ln(c_1) + \lambda \left( Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

The first-order conditions are

$$\frac{1}{c_0^*} - \lambda^* = 0$$

for $c_0$ and

$$\frac{\beta}{c_1^*} - \frac{\lambda^*}{1+r} = 0$$

for $c_1$.

b. With $\beta = 4/5$ and $1 + r = 5/4$, the first-order conditions from part (a) imply that

$$c_0^* = \frac{1}{\lambda^*}$$

and

$$c_1^* = \frac{1}{\lambda^*},$$

and with $Y_0 = 80$, and $Y_1 = 125$, the binding constraint becomes

$$80 + \frac{4 \times 125}{5} = 180 = c_0^* + \frac{4 \times c_1^*}{5}.$$ 

Although there are many ways of solving this system of three equations in three unknowns, one is to substitute $c_0^* = c_1^* = 1/\lambda^*$ into the budget constraint to obtain

$$180 = \frac{1}{\lambda^*} \left( 1 + \frac{4}{5} \right) = \frac{1}{\lambda^*} \left( \frac{9}{5} \right).$$
or

\[
\frac{1}{\lambda^*} = \frac{180 \times 5}{9} = 100,
\]

and then to substitute this solution for \(1/\lambda^*\) back into the first-order conditions to find \(c_0^* = 100\) and \(c_1^* = 100\) for optimal consumptions.

c. With \(\beta = 4/5\) and \(1 + r = 5/4\), the first-order conditions continue to imply that

\[
c_0^* = \frac{1}{\lambda^*} \text{ and } c_1^* = \frac{1}{\lambda^*}.
\]

With \(Y_0 = 180\) and \(Y_1 = 0\), the binding constraint is

\[
180 = c_0^* + \frac{4 \times c_1^*}{5}.
\]

Interestingly, the system of three equations in this case is exactly the same as it was, above, for part (b). The solutions

\[
c_0^* = 100 \text{ and } c_1^* = 100
\]

for optimal consumptions will therefore be exactly the same as well. The reason is that for this consumer, all that matters is the present discounted value of income

\[
Y_0 + \frac{Y_1}{1+r},
\]

which stays the same across the two specific examples in parts (b) and (c). Exactly how this present value breaks down into income received in each of the two periods does not matter, because the consumer can save or borrow as necessary to rearrange consumption across the two periods optimally.

2. Implementing State-Contingent Consumption Plans

In this example, contingent claims for the good and bad state at \(t = 1\) both trade for the price of one at \(t = 0\). To answer each part of this question, it is helpful to tabulate the cash flows from long and short positions in each claim as follows:

<table>
<thead>
<tr>
<th>Investment Strategy</th>
<th>Cash Flow at (t = 0)</th>
<th>Cash Flows at (t = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Good State</td>
<td>Bad State</td>
</tr>
<tr>
<td>Buy a Claim for the Good State</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Buy a Claim for the Bad State</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>Sell Short a Claim for the Good State</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Sell Short a Claim for the Bad State</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
a. To increase consumption by one unit in the bad state at $t = 1$, the consumer should buy a claim for the bad state. This requires him or her to give up one unit of consumption at $t = 0$.

b. If the consumer sells short a contingent claim for the good state, he or she can increase consumption by one unit at $t = 0$, while giving up one unit of consumption in the good state at $t = 1$.

c. Suppose the consumer sells short a contingent claim for good state and uses the proceeds of the sale to buy a contingent claim for the bad state. This strategy allows the consumer to increase his or her consumption by one unit in the bad state at $t = 1$, while decreasing consumption by one unit in the good state at $t = 1$ and keeping unchanged consumption at $t = 0$.

3. Stocks, Bonds, and Stock Options

The stock sells for $q^s = 2.10$ at $t = 0$, $P^G = 4$ in the good state at $t = 1$, and $P^B = 1$ in the bad state at $t = 1$. The bond sells for $q^b = 0.90$ at $t = 0$ and pays off 1 in both the good and bad states at $t = 1$.

a. In the good state at $t = 1$, the call option with strike price $K = 2$ allows the holder to buy the stock for 2 and immediately sell it for $P^G = 4$. In this case, it is optimal for the holder to exercise the option and collect the payoff of 2. In the bad state, the call option allows the holder to buy the stock for 2, but the market price is only $P^B = 1$. In this case, it is optimal for the holder to leave the option unexercised; the payoff is therefore zero.

b. Following Merton, we want to form a portfolio consisting of $s$ shares of stock and $b$ bonds to replicate the payoffs from the stock option. In the good state, this requires the payoff from the portfolio to equal 2. Since the stock price is $P^G = 4$ and the bond’s payoff is one, this requirement is summarized by the equation

$$2 = 4s + b.$$  

In the bad state, the payoff from the portfolio should equal zero. Since the stock price is $P^B = 1$ and the bond’s payoff is one, this requirement is summarized by

$$0 = s + b.$$  

We now have two equations in two unknowns, which can be solved by elimination or substitution. Taking the latter approach, rewrite the second equation as $b = -s$ and substitute it into the first equation to get

$$2 = 4s + b = 4s - s = 3s.$$  

From these results, we can see that the portfolio that replicates the payoffs on the option will consist of $s = 2/3$ shares of stock and $b = -2/3$ bonds. This portfolio can
be interpreted as one that takes a long position in the stock and a short position in bonds or, since selling bonds short is equivalent to borrowing, one that involves buying stock on margin.

c. No arbitrage requires that the price of the option equal the cost of assembling the portfolio identified in part (b), above. Since the stock sells for $q^s = 2.10$ and the bond for $q^b = 0.90$,

$$q^o = (2/3)q^s - (2/3)q^b = (2/3)(2.10 - 0.90) = (2/3)(1.20) = 0.80.$$

4. Pricing Riskless Cash Flows

A one-year discount bond sells for $P_1 = 0.90$ today and pays off one dollar, for sure, one year from now. A two-year discount bond sells for $P_2 = 0.60$ today and pays off one dollar, for sure, two years from now. A three-year discount bond sells for $P_3 = 0.50$ today and pays off one dollar, for sure, three years from now.

a. The payoffs on a coupon bond that makes annual interest payments of $10 each year, every year, for the next three years and also returns face value of $100 three years from now can be replicated by a portfolio that consists of ten one-year discount bonds, ten two-year discount bonds, and 110 three-year discount bonds. If there are no arbitrage opportunities across markets for coupon and discount bonds, the price $P^C$ of the coupon bond must equal the cost of assembling the portfolio of discount bonds. Therefore,

$$P^C = 10P_1 + 10P_2 + 110P_3 = 9.00 + 6.00 + 55.00 = 70.00.$$

b. The payoffs on a risk-free asset that pays off $100 two years from now and $500 three years from now can be replicated by a portfolio that consists of 100 two-year discount bonds and 500 three-year discount bonds. If there are no arbitrage opportunities across markets for risk-free assets, the price $P^A$ of this new asset must equal the cost of assembling the portfolio of discount bonds. Therefore,

$$P^A = 100P_2 + 500P_3 = 60 + 250 = 310.00.$$

c. This last part of the problem asks you to identify the portfolio of discount bonds that replicates the payoffs made by a one-year loan of $1 made one year from now and paid back with interest two years from now. Buying a one-year discount bond provides you with the $1 you want to receive one year from now, but costs $0.90 today. To offset this cost, however, you can sell short $0.90/$0.60 = 1.5 two-year discount bonds today; since each of these bonds sells for $0.60 today, this will provide you with the $1.5 \times $0.60 = $0.90 that you need to do so. The short position in two-year discount bonds, however, requires you to pay $1.50 two years from now. The implied two-year forward rate equals 50 percent.
5. Criteria for Choosing Between Risky Alternatives

Two assets have percentage returns $\tilde{R}_1$ and $\tilde{R}_2$ that vary across two states as follows:

<table>
<thead>
<tr>
<th>Return on</th>
<th>State 1</th>
<th>State 2</th>
<th>$E(\tilde{R})$</th>
<th>$\sigma(\tilde{R})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>$\tilde{R}_1$</td>
<td>6</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Asset 2</td>
<td>$\tilde{R}_2$</td>
<td>9</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

a. There is no state-by-state dominance, because asset 2 has the higher return in the good state but asset 1 has the higher return in the bad state.

b. There is no mean-variance dominance either, because asset 2 has a higher expected return but asset 1’s return has lower standard deviation.

c. If a third asset becomes available that has percentage return equal to 10 in the good state and 4 in the bad, then it will be chosen over assets 1 and 2 by any investor who prefers more to less. This is because asset three exhibits state-by-state dominance over the other two: it always provides at least as much and sometimes pays off more.
This exam has five questions on four pages; before you begin, please check to make sure that your copy has all five questions and all four pages. The five questions will be weighted equally in determining your overall exam score.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

1. Expected Utility, Risky Assets, and Certainty Equivalents

Suppose that a risk-averse investor with von Neumann-Morgenstern expected utility and initial income $Y_0 = 9$ is offered a risky asset with random payoff $\tilde{Z}$ that equals 7 in a good state that occurs with probability 1/2 and 0 in a bad state that occurs with probability 1/2.

a. If this investor is offered a choice between the risky asset described above or a safe asset that pays off 3.5 no matter what, which one will he or she take: the risky asset or the safe asset?

b. Recall that the certainty equivalent for $\tilde{Z}$ is defined as the amount $CE(\tilde{Z})$ that makes the investor indifferent between taking the risky asset and taking a safe asset that pays off $CE(\tilde{Z})$ for sure. Will $CE(\tilde{Z})$ be greater than, less than, or equal to 3.5?

c. Now suppose that the investor has Bernoulli utility function of the form

$$u(Y) = Y^{1/2},$$

where $Y$ is his or her income in any given state and, as you will recall, raising $Y$ to the power 1/2 means taking the square root of $Y$. Use the information given above to calculate the numerical value of the certainty equivalent $CE(\tilde{Z})$ for the risky asset $\tilde{Z}$.
2. Risk Aversion and Portfolio Allocation

Consider the portfolio allocation problem faced by an investor who has initial wealth $Y_0 = 100$. The investor allocates the amount $a$ to stocks, which provide a return of $r_G = 0.15$ (15 percent) in a good state that occurs with probability $\pi = 0.80 = 4/5$ and a return of $r_B = -0.05$ (-5 percent) in a bad state that occurs with probability $1 - \pi = 0.20 = 1/5$. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides the return $r_f = 0.05$ (5 percent) in both states. If the investor has von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma},$$

his or her portfolio allocation problem can be stated mathematically as

$$\max_a \pi \left\{ \frac{[(1 + r_f)Y_0 + a(r_G - r_f)]^{1-\gamma} - 1}{1 - \gamma} \right\} + (1 - \pi) \left\{ \frac{[(1 + r_f)Y_0 + a(r_B - r_f)]^{1-\gamma} - 1}{1 - \gamma} \right\}.$$ 

a. Suppose that the parameter in the investor’s Bernoulli utility function is $\gamma = 2$; this choice implies that the investor has a constant coefficient of relative risk aversion equal to 2. Use this setting for $\gamma$, together with the values of the other parameters given above, to find the numerical value of the investor’s optimal choice $a^*$. Note: Recall from our discussions in class that there are two ways of finding $a^*$. One is to substitute the parameter values into the investor’s problem and then take the first-order condition for the optimal choice of $a$, and the other is to take the first-order condition for the optimal choice of $a$ and then substitute the parameter values into this first-order condition. Use whichever approach seems easiest to you; either way, you’ll get the same answer for $a^*$.

b. Suppose that, instead of $\gamma = 2$, the investor’s constant coefficient of relative risk aversion is $\gamma = 4$. Would the value of the investor’s optimal choice $a^*$ with this larger value of $\gamma$ be greater than, less than, or the same as the value of $a^*$ that you found for part (a), above? Note: To answer this part of the problem, you don’t have to actually calculate the new value of $a^*$, all you need to do is say whether it is greater than, less than, or the same as the value you obtained in part (a).

c. Go back to assuming that $\gamma = 2$, but suppose now that the investor’s Bernoulli utility function takes the form

$$u(Y) = 2 \left[ \frac{Y^{1-\gamma} - 1}{1 - \gamma} \right].$$

With this slightly different Bernoulli utility function, would the value of the investor’s optimal choice $a^*$ be greater than, less than, or the same as the value of $a^*$ that you found for part (a), above? Note: To answer this part of the problem, you don’t have to actually calculate the new value of $a^*$, all you need to do is say whether it is greater than, less than, or the same as the value you obtained in part (a).
3. The Gains from Diversification

Consider an investor who forms a portfolio of two risky assets by allocating the share \( w = 1/2 \) of his or her initial wealth to risky asset 1, with expected return \( \mu_1 = 8 \) and standard deviation of its risky return equal to \( \sigma_1 = 2 \), and allocating the remaining share \( 1 - w = 1/2 \) of his or her initial wealth to risky asset 2, with expected return \( \mu_2 = 4 \) and standard deviation of its risky return equal to \( \sigma_2 = 2 \).

a. What will the expected return \( \mu_P \) and the standard deviation \( \sigma_P \) of the return on the investor’s portfolio be if the correlation between the two asset’s returns is \( \rho_{12} = 0 \)?

b. What will the expected return \( \mu_P \) and the standard deviation \( \sigma_P \) of the return on the investor’s portfolio be if the correlation between the two asset’s returns is \( \rho_{12} = 1 \)?

c. What will the expected return \( \mu_P \) and the standard deviation \( \sigma_P \) of the return on the investor’s portfolio be if the correlation between the two asset’s returns is \( \rho_{12} = -1 \)?

4. Portfolio Allocation with Mean-Variance Utility

Consider an investor whose preferences are described by a utility function defined directly over the mean \( \mu_P \) and variance \( \sigma_P^2 \) of the random return that he or she earns from his or her portfolio; suppose, in particular, that this utility function takes the form

\[
U(\mu_P, \sigma_P^2) = \mu_P - \left( \frac{A}{2} \right) \sigma_P^2.
\]

The investor creates this portfolio by allocating the fraction \( w_1 \) of his or her initial wealth to risky asset 1, with expected return \( E(\tilde{r}_1) \) and variance of its risky return \( \sigma_1^2 \), fraction \( w_2 \) to risky asset 2, with expected return \( E(\tilde{r}_2) \) and variance of its risky return \( \sigma_2^2 \), and the remaining fraction \( 1 - w_1 - w_2 \) to risk-free assets with return \( r_f \). The mean or expected return on the investor’s portfolio is therefore

\[
\mu_P = (1 - w_1 - w_2)r_f + w_1E(\tilde{r}_1) + w_2E(\tilde{r}_2)
\]

and, under the additional assumption that the correlation between the random returns on the two risky assets is \( \rho_{12} = 0 \), the variance of the return on the investor’s portfolio is

\[
\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2.
\]

Thus, the investor solves the portfolio allocation problem

\[
\max_{w_1, w_2} (1 - w_1 - w_2)r_f + w_1E(\tilde{r}_1) + w_2E(\tilde{r}_2) - \left( \frac{A}{2} \right) (w_1^2\sigma_1^2 + w_2^2\sigma_2^2).
\]

a. Write down the first-order conditions that determine the investor’s optimal choices \( w_1^* \) and \( w_2^* \) for the two portfolio shares \( w_1 \) and \( w_2 \).
b. Suppose, in particular, that \( E(\tilde{r}_1) = 10, \sigma_1^2 = 16, E(\tilde{r}_2) = 6, \sigma_2^2 = 16, r_f = 2, \text{ and } A = 1 \). What are the optimal choices \( w_1^*, w_2^* \), and \( 1 - w_1^* - w_2^* \) in this case?

c. Suppose that the value of \( A \) in this problem was equal to 2 instead of 1. Would you expect the investor to allocate a larger or a smaller share \( 1 - w_1^* - w_2^* \) of his or her initial wealth to risk-free assets? *Note:* To answer this part of the problem, you don’t have to actually calculate the new value of \( 1 - w_1^* - w_2^* \), all you need to do is to say whether it is larger or smaller than the answer you obtained in part (b), above.

### 5. The Capital Asset Pricing Model

According to the capital asset pricing model (CAPM), the expected return \( E(\tilde{r}_j) \) on any risky asset \( j \) is related to the risk-free rate \( r_f \) and the expected return \( E(\tilde{r}_M) \) on the market portfolio according to

\[
E(\tilde{r}_j) = r_f + \beta_j [E(\tilde{r}_M) - r_f],
\]

where \( \beta_j \), risky asset \( j \)'s “beta,” is calculated as

\[
\beta_j = \frac{\sigma_{jM}}{\sigma_M^2},
\]

where \( \sigma_{jM} \) is the covariance between the random return \( \tilde{r}_j \) and the random return \( \tilde{r}_M \) on the market portfolio and \( \sigma_M^2 \) is the variance of the the random return \( \tilde{r}_M \) on the market portfolio. With this relationship in mind, please indicate whether each of the following statements are true or false. *Note:* All you need to do for this question is to indicate whether each statement is true or false; you don’t need to explain why.

a. According to the capital asset pricing model, if two individual stocks – asset 1 and asset 2 – have the same beta, so that \( \beta_1 = \beta_2 \), then these two stocks must have the same expected returns, even if the variance \( \sigma_1^2 \) of asset 1’s random return \( \tilde{r}_1 \) is larger than the variance \( \sigma_2^2 \) of asset 2’s random return \( \tilde{r}_2 \).

b. According to the capital asset pricing model, if an individual stock – asset 3 – has a random return \( \tilde{r}_3 \) that is uncorrelated with the the random return \( \tilde{r}_M \) on the market portfolio, then asset 3’s expected return will equal the risk-free rate \( r_f \).

c. According to the capital asset pricing model, if the expected return on the market portfolio is higher than the risk-free rate, then any stock with a negative beta will have an expected return that is less than the risk-free rate.
1. Expected Utility, Risky Assets, and Certainty Equivalents

A risk-averse investor with von Neumann-Morgenstern expected utility and initial income $Y_0 = 9$ is offered a risky asset with random payoff $\tilde{Z}$ that equals 7 in a good state that occurs with probability 1/2 and 0 in a bad state that occurs with probability 1/2.

a. Since expected payoff on the risky asset is $E(\tilde{Z}) = 3.5$, the investor will always choose a safe asset that pays 3.5 for sure over the risky asset.

b. Since a risk-averse investor will accept an amount less than $E(\tilde{Z}) = 3.5$ to avoid the risk associated with the random payoff $\tilde{Z}$, $CE(\tilde{Z})$ will always be less than 3.5.

c. Assuming that the investor has Bernoulli utility function of the form

$$u(Y) = Y^{1/2},$$

the certainty equivalent $CE(\tilde{Z})$ must satisfy

$$[9 + CE(\tilde{Z})]^{1/2} = (1/2)(9 + 7)^{1/2} + (1/2)(9)^{1/2},$$

where the left-hand side measures the utility from getting $CE(\tilde{Z})$ for sure and the right-hand side measures the expected utility from the random payoff $\tilde{Z}$ that equals 7 with probability 1/2 and 0 with probability 1/2. Taking the square roots on the right-hand size yields

$$[9 + CE(\tilde{Z})]^{1/2} = (1/2)(16)^{1/2} + (1/2)(9)^{1/2} = (1/2)4 + (1/2)3 = 7/2.$$  

Squaring both sizes then implies

$$9 + CE(\tilde{Z}) = 49/4$$

Finally, subtracting 9 from both sides provides the solution

$$CE(\tilde{Z}) = 49/4 - 36/4 = 13/4 = 3.25,$$

an answer that confirms that $CE(\tilde{Z}) < E(\tilde{Z})$.  

2. Risk Aversion and Portfolio Allocation

An investor who has initial wealth $Y_0 = 100$ allocates the amount $a$ to stocks, which provide a return of $r_G = 0.15$ (15 percent) in a good state that occurs with probability $\pi = 0.80 = 4/5$ and a return of $r_B = -0.05$ (−5 percent) in a bad state that occurs with probability $1 - \pi = 0.20 = 1/5$. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides the return $r_f = 0.05$ (5 percent) in both states. If the investor has von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma},$$

his or her portfolio allocation problem can be stated mathematically as

$$\max_a \pi \left\{ \frac{[(1 + r_f)Y_0 + a(r_G - r_f)]^{1-\gamma} - 1}{1-\gamma} \right\} + (1 - \pi) \left\{ \frac{[(1 + r_f)Y_0 + a(r_B - r_f)]^{1-\gamma} - 1}{1-\gamma} \right\}.$$

a. The first-order condition for the general problem stated above is

$$\pi (r_G - r_f) \left[ (1 + r_f)Y_0 + a^*(r_G - r_f) \right]^{(1-\gamma)/\gamma} + (1 - \pi) (r_B - r_f) \left[ (1 + r_f)Y_0 + a^*(r_B - r_f) \right]^{(1-\gamma)/\gamma} = 0.$$

When $\gamma = 2$ and the other parameters are set as indicated above, this first-order condition specializes to

$$\frac{(4/5)(0.10)}{(105 + 0.10a^*)^2} + \frac{(1/5)(-0.10)}{(105 - 0.10a^*)^2} = 0.$$

Move the second term on the left-hand side over to the right to get rid of the minus sign; then, multiply both sides by 5 and divide both sides by 0.10 to obtain

$$\frac{4}{(105 + 0.10a^*)^2} = \frac{1}{(105 - 0.10a^*)^2}.$$

Next, take the square root of both sizes to get

$$\frac{2}{105 + 0.10a^*} = \frac{1}{105 - 0.10a^*}.$$

This last equation leads to the solution for $a^*$ using the final steps

$$2(105 - 0.10a^*) = 105 + 0.10a^*$$

$$210 - 0.20a^* = 105 + 0.10a^*$$

$$105 = 0.30a^*$$

$$a^* = \frac{105}{0.30} = 350.$$
b. When $\gamma = 4$, the investor is more risk averse than when $\gamma = 2$. Therefore, $a^*$ in this case will be less than the solution from part (a).

c. Multiplying the Bernoulli utility function by the positive constant 2, to obtain

$$u(Y) = 2 \left[ \frac{Y^{1-\gamma} - 1}{1-\gamma} \right].$$

does not change the preference ordering described by the original utility function used for part (a). Therefore, $a^*$ in this case will be the same as the solution from part (a).

3. The Gains from Diversification

If an investor forms a portfolio of two risky assets by allocating the share $w = 1/2$ of his or her initial wealth to risky asset 1, with expected return $\mu_1 = 8$ and standard deviation of its risky return equal to $\sigma_1 = 2$, and allocating the remaining share $1 - w = 1/2$ of his or her initial wealth to risky asset 2, with expected return $\mu_2 = 4$ and standard deviation of its risky return equal to $\sigma_2 = 2$, then the expected return on the portfolio is

$$\mu_P = w\mu_1 + (1 - w)\mu_2 = (1/2)8 + (1/2)4 = 4 + 2 = 6$$

and the standard deviation of the return on the portfolio is

$$\sigma_P = \sqrt{w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}}^{1/2}$$

$$= \sqrt{[(1/4)4 + (1/4)4 + 2(1/2)(1/2)4\rho_{12}]}^{1/2}$$

$$= \sqrt{(2 + 2\rho_{12})^{1/2}},$$

where $\rho_{12}$ is the correlation between the two asset’s returns.

a. The formulas from above imply that if $\rho_{12} = 0$, then $\mu_P = 6$ and $\sigma_P = \sqrt{2} = 1.414$.

b. If $\rho_{12} = 1$, then $\mu_P = 6$ and $\sigma_P = \sqrt{4} = 2$.

c. If $\rho_{12} = -1$, then $\mu_P = 6$ and $\sigma_P = 0$.

4. Portfolio Allocation with Mean-Variance Utility

An investor solves the portfolio allocation problem

$$\max_{w_1, w_2} (1 - w_1 - w_2)r_f + w_1E(\tilde{r}_1) + w_2E(\tilde{r}_2) - \left( \frac{A}{2} \right) (w_1^2\sigma_1^2 + w_2^2\sigma_2^2).$$

a. The first-order condition for $w_1$ is

$$E(\tilde{r}_1) - r_f - A\sigma_1^2w_1^* = 0$$

and the first-order condition for $w_2$ is

$$E(\tilde{r}_2) - r_f - A\sigma_2^2w_2^* = 0.$$
b. With $E(\tilde{r}_1) = 10$, $\sigma_1^2 = 16$, $E(\tilde{r}_2) = 6$, $\sigma_2^2 = 16$, $r_f = 2$, and $A = 1$, the first-order conditions imply that

$$10 - 2 - 16w_1^* = 0$$

and

$$6 - 2 - 16w_2^* = 0$$

or

$$w_1^* = 8/16 = 1/2$$

and

$$w_2^* = 4/16 = 1/4.$$  

c. An investor with $A = 2$ is more risk averse than an investor with $A = 1$. This investor would therefore allocate a larger share of his or her initial wealth to risk-free assets.

5. The Capital Asset Pricing Model

According to the capital asset pricing model (CAPM), the expected return $E(\tilde{r}_j)$ on any risky asset $j$ is related to the risk-free rate $r_f$ and the expected return $E(\tilde{r}_M)$ on the market portfolio according to

$$E(\tilde{r}_j) = r_f + \beta_j [E(\tilde{r}_M) - r_f],$$

where $\beta_j$, risky asset $j$'s “beta,” is calculated as

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2},$$

where $\sigma_{jM}$ is the covariance between the random return $\tilde{r}_j$ and the random return $\tilde{r}_M$ on the market portfolio and $\sigma_M^2$ is the variance of the the random return $\tilde{r}_M$ on the market portfolio.

a. This statement is true: According to the capital asset pricing model, if two individual stocks – asset 1 and asset 2 – have the same beta, so that $\beta_1 = \beta_2$, then these two stocks must have the same expected returns, even if the variance $\sigma_1^2$ of asset 1’s random return $\tilde{r}_1$ is larger than the variance $\sigma_2^2$ of asset 2’s random return $\tilde{r}_2$.

b. This statement is also true: According to the capital asset pricing model, if an individual stock – asset 3 – has a random return $\tilde{r}_3$ that is uncorrelated with the the random return $\tilde{r}_M$ on the market portfolio, then asset 3’s expected return will equal the risk-free rate $r_f$.

c. And this statement is true as well: According to the capital asset pricing model, if the expected return on the market portfolio is higher than the risk-free rate, then any stock with a negative beta will have an expected return that is less than the risk-free rate.
1. Unconstrained Optimization

Suppose that an automobile manufacturer is able to produce \( c \) cars at total cost \( \alpha c^2 \), where \( \alpha > 0 \) is a positive parameter. If the manufacturer sells each car at price \( p > 0 \) in a perfectly competitive market, its profits are \( pc - \alpha c^2 \).

a. Consider the firm’s profit maximization problem

\[
\max_c pc - \alpha c^2.
\]

Since the objective function is concave, the first-order condition obtained by differentiating the objective function by the choice variable \( c \) and equating to zero is both a necessary and sufficient condition for the value \( c^* \) that solves this problem. Write down this first-order condition.

b. Next, re-arrange the first-order condition to get an equation that shows how \( c^* \) depends on the output price \( p \) and the cost parameter \( \alpha \).

c. Finally, use your solution from part (b) to answer the following two questions. First, what happens to the firm’s optimal choice of \( c^* \) when the output price \( p \) rises? Second, what happens to \( c^* \) when the cost parameter \( \alpha \) falls?
2. Consumer Optimization

Consider a consumer who purchases $c_0$, $c_1$, and $c_2$ of three goods – goods 0, 1, and 2 – in perfectly competitive markets at the prices $p_0$, $p_1$, and $p_2$ in order to maximize the utility function

$$\ln(c_0) + \ln(c_1) + \ln(c_2)$$

subject to the budget constraint

$$Y \geq p_0 c_0 + p_1 c_1 + p_2 c_2,$$

where $Y$ denotes the consumer’s income and $\ln$ is the natural logarithm function.

a. Set up the Lagrangian for the consumer’s problem: choose $c_0$, $c_1$, and $c_2$ to maximize the utility function subject to the budget constraint. Then, write down the first-order conditions for $c_0$, $c_1$, and $c_2$ that characterize the solution to this problem.

b. Because the natural log function is strictly increasing, it will always be optimal for the consumer to spend all of his or her income. That is, the budget constraint will hold as the equality

$$Y = p_0 c_0^* + p_1 c_1^* + p_2 c_2^*,$$

when $c_0^*$, $c_1^*$, and $c_2^*$ are chosen optimally. Use this binding budget constraint, together with your first-order conditions from part (a), above, to obtain solutions that show how $c_0^*$, $c_1^*$, and $c_2^*$ depend on the prices $p_0$, $p_1$, and $p_2$ and the consumer’s income $Y$.

c. Finally, use your results from part (b), above, to answer the following question: what fraction of his or her total income $Y$ does the consumer spend on each good 0, 1, and 2?

3. Intertemporal Consumer Decision-Making

The figure on the next page illustrates the tangency condition describing optimal choices in Irving Fisher’s model of intertemporal consumer decision-making. The graph measures consumption today, $c_0$, along the horizontal axis and consumption next year, $c_1$, along the vertical axis. The slope of the straight line, the intertemporal budget constraint, is $-(1+r)$, where $r$ denotes the interest rate at which the consumer can save or borrow. The slope of the curved line, the indifference curve, is the intertemporal marginal rate of substitution, which equals $u'(c_0)/\beta u'(c_1)$ when the consumer’s utility function takes the general form

$$u(c_0) + \beta u(c_1)$$

and the discount factor $\beta$ is a measure of the consumer’s patience: more patient consumers have higher values of $\beta$. 

2
In this graph, the consumer’s optimal choices $c^*_0$ and $c^*_1$ are found at the point of tangency between the indifference curve and budget constraint, where

$$-\frac{u'(c^*_0)}{\beta u'(c^*_1)} = -(1 + r).$$

a. Suppose that the consumer’s utility function takes the more specific form

$$\ln(c_0) + \beta \ln(c_1),$$

where $\ln$ is the natural logarithm. Use the optimality condition from above, together with this specific choice for the utility function, to derive an expression linking the optimal consumption growth rate $c^*_1/c^*_0$ to the discount factor $\beta$ and the interest rate $r$.

b. Next, use your answer from part (a) to answer the question: Do more patient consumers choose faster or slower rates of consumption growth? That is, does $c^*_1/c^*_0$ go up or down when $\beta$ rises?

c. Finally, use your answer from part (a) to answer the question: Do higher interest rates lead consumers to choose faster or slower rates of consumption growth? That is, does $c^*_1/c^*_0$ go up or down when $r$ rises?
4. Stocks, Bonds, Contingent Claims, and Stock Options

Consider an economy in which there are two periods $t = 0$ (today) and $t = 1$ (next year) and two states at $t = 1$: a good and bad state that occur with equal probabilities $\pi = 1 - \pi = 1/2$.

Suppose that, in this economy, two assets are traded. A risky stock sells for $q^s = 1$ at $t = 0$, $P^G = 3$ in the good state at $t = 1$, and $P^B = 1$ in the bad state at $t = 1$. A risk-free bond sells for $q^b = 0.60$ at $t = 0$ and pays off 1 in both the good and bad states at $t = 1$.

a. Find the combination of purchases and/or short sales of shares of stock and bonds that will replicate the payoffs on a contingent claim for the good state at $t = 1$. Then, find the price at which the contingent claim for the good state should trade if there are to be no arbitrage opportunities across the markets for stocks, bonds, and contingent claims.

b. Find the combination of purchases and/or short sales of shares of stock and bonds that will replicate the payoffs on a contingent claim for the bad state at $t = 1$. Then, find the price at which the contingent claim for the bad state should trade if there are to be no arbitrage opportunities across the markets for stocks, bonds, and contingent claims.

c. Consider a put option, which gives the holder the right, but not the obligation, to sell a share of the stock at the strike price $K = 2$ at $t = 1$. In the good state at $t = 1$, the share trades for $P^G = 3$, so it will not be worthwhile for the option holder to exercise his or her right to sell at the lower strike price $K = 2$. In the bad state, however, the option holder can buy a share of stock for $P^B = 1$ and, by exercising the option, simultaneously sell the share of stock at the higher strike price $K = 2$, thereby earning a profit of 1. The put option, therefore, has a payoff of zero in the good state and 1 in the bad state at $t = 1$. Use this information to find the price, $q^p$, at which the put option should trade at $t = 0$, if there are to be no arbitrage opportunities across the markets for stocks, bonds, contingent claims, and stock options.
5. Pricing Riskless Cash Flows

Consider an economy in which, initially, two risk-free discount bonds and one risk-free coupon bond are traded. Specifically, a one-year discount bond sells for $P_1 = 0.75$ today and pays off one dollar, for sure, one year from now. A two-year discount bond sells for $P_2 = 0.50$ today and pays off one dollar, for sure, two years from now. Finally, a three-year coupon bond sells for $P_3^C = 1.85$ today and makes annual interest (coupon) payments of one dollar at the end each and every one of the next three years and makes an additional payment of face value of one dollar at the end of the third year.

a. Suppose now that a risk-free, two-year coupon bond is introduced into this economy. This new coupon bond makes annual interest (coupon) payments of one dollar at the end of each and every one of the next two years and also makes a payment of face value of one dollar at the end of the second year. At what price $P_2^C$ will this new, two-year coupon bond sell for today, if there are no arbitrage opportunities across all markets for risk-free assets?

b. Suppose, next, that another new risk-free asset is introduced, which makes two payments: one dollar one year from now and one dollar two years from now. At what price $P^A$ will this risk-free asset sell for today, if there are no arbitrage opportunities across all markets for risk-free assets?

c. Suppose, finally, that yet another new a risk-free asset if introduced, which pays off $2$ for sure, three years from now. At what price $P^B$ will this risk-free asset sell for today, if there are no arbitrage opportunities across all markets for risk-free assets?
1. Unconstrained Optimization

The automobile manufacturer solves

$$\max_c pc - \alpha c^2.$$ 

a. The first-order condition for the value $c^*$ that solves this problem is

$$p - 2\alpha c^* = 0.$$ 

b. Moving the second term on the left-hand side of this first-order condition over to the right and dividing through by $2\alpha$ yields the solution

$$c^* = \frac{p}{2\alpha}.$$ 

c. The solution shows that the optimal number $c^*$ are cars produced rises when the price $p$ goes up and rises when the cost parameter $\alpha$ falls.

2. Consumer Optimization

The consumer chooses $c_0$, $c_1$, and $c_2$ to maximize the utility function

$$\ln(c_0) + \ln(c_1) + \ln(c_2)$$

subject to the budget constraint

$$Y \geq p_0 c_0 + p_1 c_1 + p_2 c_2.$$ 

a. The Lagrangian for the consumer’s problem is

$$L(c_0, c_1, c_2, \lambda) = \ln(c_0) + \ln(c_1) + \ln(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$ 

The first-order conditions, obtained by differentiating the Lagrangian by each choice variable and setting the result equal to zero, are

$$\frac{1}{c_0^*} - \lambda^* p_0 = 0,$$

$$\frac{1}{c_1^*} - \lambda^* p_1 = 0,$$

and

$$\frac{1}{c_2^*} - \lambda^* p_2 = 0.$$
b. When combined with the binding budget constraint

\[ Y = p_0 c_0^* + p_1 c_1^* + p_2 c_2^*, \]

the first-order conditions from part (a), above, form a system of 4 equations in 4 unknowns: \( c_0^*, c_1^*, c_2^*, \) and \( \lambda^* \). Although there are a variety of ways to solve these equations, probably the easiest is to rewrite the first-order conditions as

\[
c_0^* = \frac{1}{\lambda^* p_0},
\]

\[
c_1^* = \frac{1}{\lambda^* p_1},
\]

and

\[
c_2^* = \frac{1}{\lambda^* p_2},
\]

and substitute these expressions into the budget constraint to find

\[
Y = p_0 \left( \frac{1}{\lambda^* p_0} \right) + p_1 \left( \frac{1}{\lambda^* p_1} \right) + p_2 \left( \frac{1}{\lambda^* p_2} \right) = \frac{3}{\lambda^*}
\]

or

\[
\lambda^* = \frac{3}{Y}.
\]

Finally, substitute this solution for \( \lambda^* \) back into the previous expressions for \( c_0^*, c_1^*, \) and \( c_2^* \) to get

\[
c_0^* = \frac{Y}{3p_0},
\]

\[
c_1^* = \frac{Y}{3p_1},
\]

and

\[
c_2^* = \frac{Y}{3p_2}.
\]

These solutions show that the optimal amounts of each good purchased rise when income \( Y \) goes up but fall when the good’s price goes up.

c. The solutions from part (b), above, imply that the shares of income spent on the three goods are all 1/3:

\[
\frac{p_0 c_0^*}{Y} = \frac{p_1 c_1^*}{Y} = \frac{p_2 c_2^*}{Y} = \frac{1}{3}.
\]

3. Intertemporal Consumer Decision-Making

In the graph, the consumer’s optimal choices \( c_0^* \) and \( c_1^* \) are found at the point of tangency between the indifference curve and budget constraint, where

\[
- \frac{u'(c_0^*)}{\beta u'(c_1^*)} = -(1 + r).
\]
a. When the consumer’s utility takes the form specific form

\[ \ln(c_0) + \beta \ln(c_1), \]

the tangency condition becomes

\[- \frac{c_1^*}{\beta c_0^*} = -(1 + r), \]

which implies that the optimal growth rate of consumption is linked to the discount factor and the interest rate via

\[ \frac{c_1^*}{c_0^*} = \beta(1 + r). \]

b. The equation from part (a) shows that \( \frac{c_1^*}{c_0^*} \) goes up when \( \beta \) rises: this means that more patient consumers choose faster rates of consumption growth.

c. The equation from part (a) also shows that \( \frac{c_1^*}{c_0^*} \) goes up when \( r \) rises: this means that higher interest rates lead consumers to choose faster rates of consumption growth.

4. Stocks, Bonds, Contingent Claims, and Stock Options

There are two periods \( t = 0 \) (today) and \( t = 1 \) (next year) and two states at \( t = 1 \): a good and bad state that occur with equal probabilities \( \pi = 1 - \pi = 1/2 \). Two assets are traded. A risky stock sells for \( q^s = 1 \) at \( t = 0 \), \( P^G = 3 \) in the good state at \( t = 1 \), and \( P^B = 1 \) in the bad state at \( t = 1 \). A risk-free bond sells for \( q^b = 0.60 \) at \( t = 0 \) and pays off 1 in both the good and bad states at \( t = 1 \).

a. Let \( s \) be the number of shares of stock purchased (or sold short, if \( s < 0 \)) and let \( b \) be the number of bond purchased (or sold short, if \( b < 0 \)), in order to replicate the payoffs from a contingent claim for the good state. Since the claim for the good state pays off one in the good state, it must be that

\[ 1 = P^G s + b = 3s + b. \]

And since the claim for the good state pays off zero in the bad state, it must also be that

\[ 0 = P^B s + b = s + b. \]

Either elimination or substitution can be used to find the values

\[ s = 1/2 \]

and

\[ b = -1/2 \]

that satisfy both of these equations. Evidently, replicating the payoffs on the claim for the good state requires buying 1/2 share of stock while selling short 1/2 bonds. If
there are to be no arbitrage opportunities across the markets for stocks, bonds, and contingent claims, the price of a contingent claim for the good state must equal the cost of assembling this portfolio of the stock and bond. In particular, the price of the contingent claim for the good state must be

\[ q^G = \frac{1}{2} q^s - \frac{1}{2} q^b = \frac{1}{2} - \frac{1}{2}(0.60) = 0.20. \]

b. Now let \( s \) be the number of shares of stock purchased (or sold short, if \( s < 0 \)) and let \( b \) be the number of bond purchased (or sold short, if \( b < 0 \)), in order to replicate the payoffs from a contingent claim for the bad state. Since the claim for the bad state pays off zero in the good state, it must be that

\[ 0 = P^G s + b = 3s + b. \]

And since the claim for the bad state pays off one in the bad state, it must also be that

\[ 1 = P^B s + b = s + b. \]

Either elimination or substitution can be used to find the values

\[ s = -\frac{1}{2} \]

and

\[ b = \frac{3}{2} \]

that satisfy both of these equations. Evidently, replicating the payoffs on the claim for the good state requires selling short \( \frac{1}{2} \) share of stock while buying \( \frac{3}{2} \) bonds. If there are to be no arbitrage opportunities across the markets for stocks, bonds, and contingent claims, the price of a contingent claim for the bad state must equal the cost of assembling this portfolio of the stock and bond. In particular, the price of the contingent claim for the bad state must be

\[ q^B = -(\frac{1}{2})q^s + (\frac{3}{2})q^b = -(\frac{1}{2})1 + (\frac{3}{2})(0.60) = 0.40. \]

c. The put option has a payoff of zero in the good state and one in the bad state. The easiest way to infer its price, if there are no arbitrage opportunities across the markets for stocks, bonds, contingent claims, and stock options, is to observe that since the option has the same payoffs as a contingent claim for the bad state, its price must be the same as the contingent claim for the bad state:

\[ q^p = q^B = 0.40. \]

Another way to price this put option is to find the portfolio of the stock and bond that replicates its payoffs. But, from part (b) above, we already know that forming this portfolio will involve selling short \( \frac{1}{2} \) share of stock while buying \( \frac{3}{2} \) bonds at a total cost of 0.40. This approach leads to the same answer: the price of the put option must be \( q^p = 0.40 \).
5. Pricing Riskless Cash Flows

Two risk-free discount bonds and one coupon bond are initially traded. A one-year discount bond sells for $P_1 = $0.75 today and pays off one dollar, for sure, one year from now. A two-year discount bond sells for $P_2 = $0.50 today and pays off one dollar, for sure, two years from now. Meanwhile, a three-year coupon bond sells for $P^C_3 = $1.85 today and makes annual interest payments of one dollar at the end each and every one of the next three years and also makes a payment of face value of one dollar at the end of the third year.

a. A new, two-year coupon bond makes annual interest payments of one dollar at the end of each and every one of the next two years and also makes a payment of face value of one dollar at the end of the second year. The payoffs from this coupon bond can be replicated by a portfolio consisting of one one-year coupon bond and two two-year coupon bonds. Therefore, if there are no arbitrage opportunities across all markets for risk-free assets, the price $P^C_2$ of this two-year coupon bond must equal the price of buying one one-year discount bond and two two-year discount bonds:

$$P^C_2 = P_1 + 2P_2 = 0.75 + 2(0.50) = 0.75 + 1.00 = 1.75.$$  

b. Another new risk-free asset makes two payments: one dollar one year from now and one dollar two years from now. These payoffs can be replicated by buying one one-year discount bond and one two-year discount bond. Therefore, if there are no arbitrage opportunities across all markets for risk-free assets, the price $P^A$ of this risk-free asset must equal the cost of buying one one-year discount bond and one two-year discount bond:

$$P^A = P_1 + P_2 = 0.75 + 0.50 = 1.25.$$  

c. Finally, a third new risk-free asset pays off $2 three years from now. A payoff of $2, for sure, three years from can be obtained by buying a three-year coupon bond. But the coupon bond also provides payments of one dollar, for sure, at the end of the next two years. One way of forming a portfolio of existing assets to replicate the payoff on the third new asset would be to buy the three-year coupon bond, then sell short a one-year discount bond and a two-year discount bond. If there are no arbitrage opportunities across all markets for risk-free assets, the price $P^B$ of the new asset should equal the cost of assembling this portfolio:

$$P^B = P^C_3 - P_1 - P_2 = 1.85 - 0.75 - 0.50 = 0.60.$$  

Another way of forming a portfolio that replicates the payoff on the third new asset is to buy the three-year coupon bond and sell short the risk-free asset described in part (b), above, which pays off one dollar one year from now and one dollar two years from now. The cost of assembling this portfolio is

$$P^C_3 - P^A = 1.85 - 1.25 = 0.60,$$

implying again that in the absence of arbitrage opportunities, $P^B = 0.60$. 
This exam has five questions on five pages; before you begin, please check to make sure that your copy has all five questions and all five pages. The five questions will be weighted equally in determining your overall exam score.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

1. Choosing Between Risky Assets

Consider two assets, with random returns $\tilde{R}_1$ and $\tilde{R}_2$ that vary across two states that occur with equal probability as follows:

<table>
<thead>
<tr>
<th>Return on</th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_1$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Asset 1 $\tilde{R}_1$</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Asset 2 $\tilde{R}_2$</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

a. Does either asset display state-by-state dominance over the other? If so, which one?

b. Suppose that an investor who prefers more to less, is risk averse, and has preferences over risky assets that can be described by a von Neumann-Morgenstern expected utility function has to choose between assets 1 and 2. Can you say for sure which asset this investor will pick, asset 1 or asset 2, or will the choice depend on his or her specific level of risk aversion?

c. Suppose now that a third asset – asset 3 – is introduced, which has a return of $\tilde{R}_3 = 4$ in both state 1 and state 2 and is therefore risk free. Which asset – asset 1, 2 or 3 – will the investor from part (b) choose now? Can you say for sure, or will the choice depend on his or her specific level of risk aversion?
2. Insurance

Consider a consumer with initial income of 100, who faces a 1/5 (20 percent) chance of incurring a loss of 75 (which then brings his or her income down to 25). Suppose the consumer’s preferences are described by a von Neumann-Morgenstern expected utility function with Bernoulli utility function
\[ u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}, \]
where, as we saw in class, \( \gamma > 0 \) measures the consumer’s constant coefficient of relative risk aversion.

a. Assume first that the investor’s coefficient of relative risk aversion is \( \gamma = 1/2 \). Find the maximum amount \( x^* \) that the consumer will pay for an insurance policy that protects him or her fully against the loss.

b. Now assume that, in addition to the 1/5 (20 percent) chance of a loss of 75, there is also a 1/5 (20 percent) chance of an even bigger loss of 100 (leaving the consumer with no income in this very bad state). Find the maximum amount that the consumer with \( \gamma = 1/2 \) will be willing to pay for insurance against all losses now.

c. Finally, go back to the case from part (a), where there is just a 1/5 (20 percent) chance of a loss of 75. But suppose that instead of \( \gamma = 1/2 \), the consumer’s coefficient of relative risk aversion is \( \gamma = 2 \). In this case, will the value of \( x^* \) be larger than, smaller than, or equal to the value you found in part (a)? Note: To answer this question, you don’t need to compute the exact value of \( x^* \); all you need to do is say whether it’s larger than, smaller than, or equal to the value when \( \gamma = 1/2 \).
3. Expected Utility and Portfolio Allocation

Consider the portfolio allocation problem faced by an investor who has initial wealth $Y_0 = 100$. The investor allocates the amount $a$ to stocks, which provide a return of $r_G = 0.35$ (35 percent) in a good state that occurs with probability $\pi = 0.50$ and a return of $r_B = -0.10$ (−10 percent) in a bad state that occurs with probability $1 - \pi = 0.50$. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides the return $r_f = 0.10$ (10 percent) in both states. If the investor has von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the form

$$u(Y) = \ln(Y),$$

where $\ln$ denotes the natural logarithm, his or her portfolio allocation problem can be stated mathematically as

$$\max_a \pi \ln[(1 + r_f)Y_0 + a(r_G - r_f)] + (1 - \pi) \ln[(1 + r_f)Y_0 + a(r_B - r_f)].$$

a. Find the numerical value of the investor’s optimal choice $a^*$. Note: Recall from our discussions in class that there are two ways of finding $a^*$. One is to substitute the parameter values into the investor’s problem and then take the first-order condition for the optimal choice of $a$, and the other is to take the first-order condition for the optimal choice of $a$ and then substitute the parameter values into this first-order condition. Use whichever approach seems easiest to you; either way, you’ll get the same answer for $a^*$.

b. Suppose now that instead of providing the return of $r_G = 0.35$ in the good state and $r_B = -0.10$ in the bad state, stocks provide the return $r_G = 0.30$ in the good state and $r_B = -0.05$ in the bad. Would the value of $a^*$ in this case be larger than, smaller than, or the same as the value of $a^*$ that you found for part (a)? Note: To answer this part of the problem, you don’t have to actually calculate the new value of $a^*$, all you need to do is say whether it is greater than, less than, or the same as the value you obtained in part (a).

c. Now go back to the original case, where stocks provide the return of $r_G = 0.35$ in the good state and $r_B = -0.10$ in the bad state, but assume that the investor’s Bernoulli utility function is

$$u(Y) = (1/2) \ln(Y).$$

Would the value of $a^*$ in this case be larger than, smaller than, or the same as the value of $a^*$ that you found for part (a)? Note: Again, to answer this part of the problem, you don’t have to actually calculate the new value of $a^*$, all you need to do is say whether it is greater than, less than, or the same as the value you obtained in part (a).
4. Portfolio Allocation and the Gains from Diversification

Consider an investor who allocates the fraction $w_1$ of his or her portfolio to asset 1, which has an expected return of $\mu_1 = 3$ and variance of its random return of $\sigma_1^2 = 3$, the fraction $w_2$ of his or her portfolio to asset 2, which has an expected return of $\mu_2 = 1$ and variance of its random return of $\sigma_2^2 = 1$, and the remaining fraction $1 - w_1 - w_2$ to asset 3, which has an expected return of $\mu_3 = 2$ and variance of its random return of $\sigma_3^2 = 2$. The investor’s portfolio, therefore, has expected return

$$\mu_p = w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 = 3w_1 + w_2 + 2(1 - w_1 - w_2)$$

and variance of its random return of

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + (1 - w_1 - w_2)^2 \sigma_3^2 = 3w_1^2 + w_2^2 + 2(1 - w_1 - w_2)^2$$

assuming that the correlations between the three asset returns are all zero. If the investor allocates all of his or her funds to asset 3 by choosing $w_1 = 0$ and $w_2 = 0$, the portfolio has an expected return of 2 and variance of its random return of 2. The investor hopes to choose the portfolio weights optimally, however, in order to minimize the variance, subject to the constraint that the expected return still equals 2. As we discussed in class, this problem can be solved by choosing $w_1$ and $w_2$ to maximize $-\sigma_p^2$ subject to the constraint that $\mu_p = 2$. Using the expressions from above, the Lagrangian for this problem is

$$L(w_1, w_2, \lambda) = -3w_1^2 - w_2^2 - 2(1 - w_1 - w_2)^2 + \lambda[3w_1 + w_2 + 2(1 - w_1 - w_2) - 2].$$

a. As a first step in solving the investor’s portfolio allocation problem, write down the first-order conditions for his or her optimal choices $w_1^*$ and $w_2^*$ of $w_1$ and $w_2$.

b. Next, use your two first-order conditions from part (a) together with the constraint

$$3w_1^* + w_2^* + 2(1 - w_1^* - w_2^*) = 2$$

to find the numerical values of $w_1^*$ and $w_2^*$ that solve the investor’s problem.

c. Finally, use your solutions from part (b) to calculate the variance of the investor’s optimal portfolio.
5. The Capital Asset Pricing Model

Let $\tilde{r}_A$, $\tilde{r}_B$, and $\tilde{r}_C$ denote random returns on three risky stocks: shares in companies A, B, and C. Likewise, let $\tilde{r}_M$ denote the random return on the stock market as a whole. Suppose these random returns have variances and covariances as follows:

<table>
<thead>
<tr>
<th></th>
<th>Random Return</th>
<th>Variance</th>
<th>Covariance with $\tilde{r}_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company A</td>
<td>$\tilde{r}_A$</td>
<td>$\sigma_A^2 = 25$</td>
<td>$\sigma_{AM} = 150$</td>
</tr>
<tr>
<td>Company B</td>
<td>$\tilde{r}_B$</td>
<td>$\sigma_B^2 = 100$</td>
<td>$\sigma_{BM} = 100$</td>
</tr>
<tr>
<td>Company C</td>
<td>$\tilde{r}_C$</td>
<td>$\sigma_C^2 = 400$</td>
<td>$\sigma_{CM} = 50$</td>
</tr>
<tr>
<td>Market</td>
<td>$\tilde{r}_M$</td>
<td>$\sigma_M^2 = 100$</td>
<td>$\sigma_{MM} = 100$</td>
</tr>
</tbody>
</table>

a. Use the information in the table to compute the CAPM “betas” for the three individual stocks and for the market as a whole.

b. According to the CAPM, will any of the individual stocks have an expected return higher than the market’s expected return $E(\tilde{r}_M)$? If so, which one(s)? According to the CAPM, will any of the individual stocks have an expected return lower than the market’s expected return $E(\tilde{r}_M)$? If so, which one(s)? Note: To answer these questions, you can assume that the market’s expected return $E(\tilde{r}_M)$ is greater than the return $r_f$ on risk-free assets.

c. According to the CAPM, will any of the individual stocks have an expected return lower than the return $r_f$ on risk-free assets? If so, which one(s)? Note: Again, to answer these questions, you can assume that the market’s expected return $E(\tilde{r}_M)$ is greater than the return $r_f$ on risk-free assets.
1. Choosing Between Risky Assets

There are two assets, with random returns $\tilde{R}_1$ and $\tilde{R}_2$ that vary across two states that occur with equal probability as follows:

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<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

a. Asset 1 displays state-by-state dominance over asset 2, because it pays off more in state 1 and no less in state 2. The numbers in the table imply that the return on asset 1 has expected value

$$E(\tilde{R}_1) = (1/2)6 + (1/2)2 = 4$$

and standard deviation

$$\sigma(\tilde{R}_1) = [(1/2)(6 - 4)^2 + (1/2)(2 - 4)^2]^{1/2} = 2,$$

while the return on asset 2 has expected value

$$E(\tilde{R}_1) = (1/2)4 + (1/2)2 = 3$$

and standard deviation

$$\sigma(\tilde{R}_1) = [(1/2)(4 - 3)^2 + (1/2)(2 - 3)^2]^{1/2} = 1.$$  

Since asset 1 has a higher mean but also a higher variance, neither asset displays mean-variance dominance over the other.

b. An investor who prefers more to less, is risk averse, and has preference over risky assets that can be described by a von Neumann-Morgenstern expected utility function will always choose an asset that displays state-by-state dominance over all others. We can therefore say for sure that this investor will choose asset 1.
c. The new asset 3 has a return \( \tilde{R}_3 = 4 \) in both state 1 and state 2. Asset 3 therefore has expected \( E(\tilde{R}_3) = 4 \) that is the same as asset 1’s expected return, but offers this expected return without any risk. The investor from part (b) will therefore prefer asset 3 to asset 1 and, since we already know that this investor prefers asset 1 to asset 2, we can say for sure that he or she will pick asset 3 over the other two.

2. Insurance

A consumer with initial income of 100 faces a 1/5 chance of incurring a loss of 75 has preferences described by a von Neumann-Morgenstern expected utility function with Bernoulli utility function

\[
u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma},\]

where \( \gamma > 0 \) measures his or her constant coefficient of relative risk aversion.

a. Assume first that the investor’s coefficient of relative risk aversion is \( \gamma = 1/2 \). The maximum amount \( x^* \) that the consumer will pay for an insurance policy that protects him or her fully against the loss can be found by finding the premium that makes the consumer indifferent between buying and not buying insurance. Thus, the value of \( x^* \) must satisfy

\[
u(100 - x^*) = (\frac{4}{5})\nu(100) + (\frac{1}{5})\nu(25),
\]

where the left-hand side measures utility with insurance and the right-hand side expected utility without. Using the specific form of the utility function together with the assumed value \( \gamma = 1/2 \) for the coefficient of relative risk aversion, this condition requires, more specifically, that

\[
\frac{(100 - x^*)^{1/2} - 1}{1/2} = (\frac{4}{5})\left(\frac{100^{1/2} - 1}{1/2}\right) + (\frac{1}{5})\left(\frac{25^{1/2} - 1}{1/2}\right).
\]

After multiplying both sides by 1/2 and adding one to both sides, this equation simplifies to

\[
(100 - x^*)^{1/2} = (\frac{4}{5})(100)^{1/2} + (\frac{1}{5})(25)^{1/2}
\]

or, even more simply,

\[
(100 - x^*)^{1/2} = (\frac{4}{5})10 + (\frac{1}{5})5 = 9.
\]

Finally, squaring both sides and rearranging yields the solution

\[
x^* = 100 - 81 = 19.
\]

b. Assuming now that, in addition to the 1/5 chance of a loss of 75, there is also a 1/5 chance of an even bigger loss of 100, the value of \( x^* \) must satisfy

\[
u(100 - x^*) = (\frac{3}{5})\nu(100) + (\frac{1}{5})\nu(25) + (\frac{1}{5})\nu(0),
\]
or, more specifically,
\[
\frac{(100 - x^*)^{1/2} - 1}{\sqrt{2}} = \left( \frac{3}{5} \right) \left( \frac{100^{1/2} - 1}{\sqrt{2}} \right) + \left( \frac{1}{5} \right) \left( \frac{25^{1/2} - 1}{\sqrt{2}} \right) + \left( \frac{1}{5} \right) \left( \frac{0^{1/2} - 1}{\sqrt{2}} \right).
\]
After multiplying both sides by \( \frac{1}{\sqrt{2}} \) and adding one to both sides, this equation simplifies to
\[
(100 - x^*)^{1/2} = \left( \frac{3}{5} \right) (100)^{1/2} + \left( \frac{1}{5} \right) (25)^{1/2} + \left( \frac{1}{5} \right) (0)^{1/2}
\]
or, even more simply,
\[
(100 - x^*)^{1/2} = \left( \frac{3}{5} \right) 10 + \left( \frac{1}{5} \right) 5 + \left( \frac{1}{5} \right) 0 = 7.
\]
As before, squaring both sides and rearranging yields the solution
\[
x^* = 100 - 49 = 51.
\]
c. A consumer with \( \gamma = 2 \) will be more risk averse than the consumer \( \gamma = 1/2 \). Therefore, we know without even solving for the exact value of \( x^* \) will be willing to pay more for insurance against the 1/5 chance of a loss of 75. If we want to confirm this, however, we can go back to the original equation defining \( x^* \),
\[
u(100 - x^*) = \left( \frac{4}{5} \right) u(100) + \left( \frac{1}{5} \right) u(25),
\]
and use the new value of \( \gamma = 2 \) to see, more specifically, that
\[
\frac{(100 - x^*)^{-1} - 1}{-1} = \left( \frac{4}{5} \right) \left( \frac{100^{-1} - 1}{-1} \right) + \left( \frac{1}{5} \right) \left( \frac{25^{-1} - 1}{-1} \right).
\]
After multiplying both sides by \(-1\) and adding 1 to both sides, this condition simplifies to
\[
(100 - x^*)^{-1} = \left( \frac{4}{5} \right) 100^{-1} + \left( \frac{1}{5} \right) 25^{-1}
\]
or
\[
\frac{1}{100 - x^*} = \frac{4}{500} + \frac{1}{125} = 0.008 + 0.008 = 0.016.
\]
Therefore, with \( \gamma = 2 \),
\[
100 - x^* = \frac{1}{0.016} = 62.5
\]
or
\[
x^* = 37.5
\]
which is much larger than the value \( x^* = 19 \) that we found, in part (a), for the consumer with \( \gamma = 1/2 \).

3. Expected Utility and Portfolio Allocation
An investor who has initial wealth \( Y_0 = 100 \) allocates the amount \( a \) to stocks, which provide a return of \( r_G = 0.35 \) in a good state that occurs with probability \( \pi = 0.50 \) and a return of
\[ r_B = -0.10 \] in a bad state that occurs with probability \(1 - \pi = 0.50\). The investor allocates the remaining amount \(Y_0 - a\) to a risk-free bond, which provides the return \(r_f = 0.10\) in both states. If the investor has von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the form

\[ u(Y) = \ln(Y), \]

his or her portfolio allocation problem can be stated mathematically as

\[
\max_a \pi \ln((1 + r_f)Y_0 + a(r_G - r_f)) + (1 - \pi) \ln((1 + r_f)Y_0 + a(r_B - r_f)).
\]

a. Using the specific values for the probabilities and returns, the investor’s problem can be restated as

\[
\max_a (1/2) \ln(110 + 0.25a) + (1/2) \ln(110 - 0.20a).
\]

The first-order condition for this problem is

\[
\frac{(1/2)(0.25)}{110 + 0.25a^*} - \frac{(1/2)(0.20)}{110 - 0.20a^*} = 0.
\]

To solve for the numerical value of \(a^*\) rewrite this condition as

\[
\frac{(1/2)(0.25)}{110 + 0.25a^*} = \frac{(1/2)(0.20)}{110 - 0.20a^*},
\]

multiply both sides by 2 to get

\[
\frac{0.25}{110 + 0.25a^*} = \frac{0.20}{110 - 0.20a^*},
\]

eliminate the fractions to obtain

\[
(0.25)110 - (0.25)(0.20)a^* = (0.20)110 + (0.20)(0.25)a^*,
\]

collect terms in \(a^*\),

\[
2(0.25)(0.20)a^* = 110(0.05)
\]

and divide to find

\[
a^* = \frac{110(0.05)}{2(0.25)(0.20)} = \frac{110}{2(5)(0.20)} = \frac{110}{2} = 55.
\]

b. If, instead of providing the return of \(r_G = 0.35\) in the good state and \(r_B = -0.10\) in the bad state, stocks provide the return \(r_G = 0.30\) in the good state and \(r_B = -0.05\) in the bad, the expected return on stocks remains the same but the volatility or riskiness of stocks goes down. Therefore, we know without even solving for the exact value of \(a^*\) that the amount allocated to stocks will go up. If we want to confirm this, however, we can use the new values for the stock returns to restate the investor’s problem as

\[
\max_a (1/2) \ln(110 + 0.20a) + (1/2) \ln(110 - 0.15a).
\]
The first-order condition for this problem is

\[ \frac{(1/2)(0.20)}{110 + 0.20a^*} - \frac{(1/2)(0.15)}{110 - 0.15a^*} = 0. \]

To solve for the numerical value of \( a^* \) rewrite this condition as

\[ \frac{(1/2)(0.20)}{110 + 0.20a^*} = \frac{(1/2)(0.15)}{110 - 0.15a^*}. \]

multiply both sides by 2 to get

\[ \frac{0.20}{110 + 0.20a^*} = \frac{0.15}{110 - 0.15a^*}, \]

eliminate the fractions to obtain

\[ (0.20)110 - (0.20)(0.15)a^* = (0.15)110 + (0.15)(0.20)a^*, \]

collect terms in \( a^* \),

\[ 2(0.20)(0.15)a^* = 110(0.05) \]

and divide to find

\[ a^* = \frac{110(0.05)}{2(0.20)(0.15)} = \frac{110}{2(4)(0.15)} = \frac{110}{1.20} = 91.67, \]

which is indeed higher than the value of \( a^* = 55 \) from part (a).

c. As we discussed in class, simply multiplying a Bernoulli utility function by a positive constant does not change the preference ordering described by the expected utility function. Therefore, if we go back to the original case, where stocks provide the return of \( r_G = 0.35 \) in the good state and \( r_B = -0.10 \) in the bad state, but assume that the investor’s Bernoulli utility function is

\[ u(Y) = \frac{1}{2} \ln(Y), \]

we will get the same value of \( a^* = 55 \) that we found in part (a). The investor’s preferences have not changed, so neither does his or her optimal portfolio allocation decision.

4. Portfolio Allocation and the Gains from Diversification

An investor allocates the fraction \( w_1 \) of his or her portfolio to asset 1, which has an expected return of \( \mu_1 = 3 \) and variance of its random return of \( \sigma_1^2 = 3 \), the fraction \( w_2 \) of his or her portfolio to asset 2, which has an expected return of \( \mu_2 = 1 \) and variance of its random return of \( \sigma_2^2 = 1 \), and the remaining fraction \( 1 - w_1 - w_2 \) to asset 3, which has an expected return of \( \mu_3 = 2 \) and variance of its random return of \( \sigma_3^2 = 2 \). The investor’s portfolio, therefore, has expected return

\[ \mu_p = w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3 = 3w_1 + w_2 + 2(1 - w_1 - w_2) \]
and variance of its random return of

\[ \sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + (1 - w_1 - w_2)^2 \sigma_3^2 = 3w_1^2 + w_2^2 + 2(1 - w_1 - w_2)^2 \]

assuming that the correlations between the three asset returns are all zero. If the investor allocates all of his or her funds to asset 3 by choosing \( w_1 = 0 \) and \( w_2 = 0 \), the portfolio has an expected return of 2 and variance of its random return of 2. The investor hopes to choose the portfolio weights optimally, however, in order to minimize the variance, subject to the constraint that the expected return still equals 2. This problem can be solved by choosing \( w_1 \) and \( w_2 \) to maximize \( -\sigma_p^2 \) subject to the constraint that \( \mu_p = 2 \). Using the expressions from above, the Lagrangian for this problem is

\[ L(w_1, w_2, \lambda) = -3w_1^2 - w_2^2 - 2(1 - w_1 - w_2)^2 + \lambda[3w_1 + w_2 + 2(1 - w_1 - w_2) - 2]. \]

a. The first-order conditions can be obtained by differentiating the Lagrangian with respect to \( w_1 \) and \( w_2 \) and, in each case, setting the result equal to zero:

\[ -6w_1^* + 4(1 - w_1^* - w_2^*) + \lambda^*(3 - 2) = 0 \]

and

\[ -2w_2^* + 4(1 - w_1^* - w_2^*) + \lambda^*(1 - 2) = 0. \]

b. Since the constraint

\[ 3w_1^* + w_2^* + 2(1 - w_1^* - w_2^*) = 2 \]

implies that

\[ w_1^* - w_2^* = 0, \]

the optimal portfolio must allocate equal shares to assets 1 and 2. Therefore, let \( w^* = w_1^* = w_2^* \), and substitute this common value for \( w_1^* \) and \( w_2^* \) into the first-order conditions from part (a), so as to write them more simply as

\[ -6w^* + 4(1 - 2w^*) + \lambda^* = 0 \]

and

\[ -2w^* + 4(1 - 2w^*) - \lambda^* = 0. \]

Eliminating \( \lambda^* \) by adding these two equations yields

\[ -8w^* + 8(1 - 2w^*) = 0 \]

or

\[ (8 + 16)w^* = 8 \]

or

\[ w^* = 1/3. \]

In this case, evidently, it is optimal for the investor to allocate equal 1/3 shares of his or her portfolio to the three assets:

\[ w_1^* = w_2^* = w_3^* = 1/3. \]
c. Substituting the optimal choices for the portfolio weights into the expression

\[ \sigma_p^2 = 3w_1^2 + w_2^2 + 2(1 - w_1 - w_2)^2 \]

for the variance of the portfolio’s random return shows that the optimal portfolio has

\[ \sigma_p^2 + (1/3)^2(3 + 1 + 2) = 6/9 = 2/3. \]

This is much smaller than the value of 2 that the investor would get simply by allocating all of his or her funds to asset 3.

5. The Capital Asset Pricing Model

Shares in companies A, B, and C and the market have random returns with variances and covariances as follows:

<table>
<thead>
<tr>
<th>Company</th>
<th>( \tilde{r} )</th>
<th>Variance ( \sigma^2 )</th>
<th>Covariance with ( \tilde{r}_M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company A</td>
<td>( \tilde{r}_A )</td>
<td>25 ( \sigma^2_A )</td>
<td>150 ( \sigma_{AM} )</td>
</tr>
<tr>
<td>Company B</td>
<td>( \tilde{r}_B )</td>
<td>100 ( \sigma^2_B )</td>
<td>100 ( \sigma_{BM} )</td>
</tr>
<tr>
<td>Company C</td>
<td>( \tilde{r}_C )</td>
<td>400 ( \sigma^2_C )</td>
<td>50 ( \sigma_{CM} )</td>
</tr>
<tr>
<td>Market</td>
<td>( \tilde{r}_M )</td>
<td>100 ( \sigma^2_M )</td>
<td>100 ( \sigma_{MM} )</td>
</tr>
</tbody>
</table>

a. In general, the CAPM beta for asset \( j \)

\[ \beta_j = \frac{\sigma_{jM}}{\sigma^2_M}, \]

where \( \sigma_{jM} \) is the covariance between asset \( j \)'s random return \( \tilde{r}_j \) and the random return on the market \( \tilde{r}_M \) and \( \sigma^2_M \) is the variance of the market’s random return. Therefore, using the data from the table, company A’s shares have

\[ \beta_A = \frac{150}{100} = 1.5, \]

company B’s shares have

\[ \beta_B = \frac{100}{100} = 1.0, \]

company C’s shares have

\[ \beta_C = \frac{50}{100} = 0.5, \]

and the market has

\[ \beta_M = \frac{100}{100} = 1.0. \]
b. According to the CAPM, the expected return $E(\tilde{r}_j)$ on asset $j$ is determined as

$$E(\tilde{r}_j) = r_f + \beta_j [E(\tilde{r}_M) - r_f],$$

where $r_f$ is the return on risk-free assets. Therefore, under the assumption that $E(\tilde{r}_M) > r_f$, any individual stock will have expected return higher than the market’s if its beta is greater than one and expected return lower than the market’s if its beta is less than one. Based on the betas computed in part (a), we can therefore say that according to the CAPM, company A’s shares will have expected return higher than the market’s and company C’s shares will have expected return less than the markets.

c. According to the CAPM, an individual stock can have an expected return lower than the return $r_f$ on risk-free assets, but only if its beta is negative. Since none on the individual stocks in the table has a negative beta, none has an expected return lower than the risk-free rate.
1. **Apple and Banana Farming**

Consider a farmer who grows and consumes $c_a$ apples and $c_b$ bananas every week. Both goods are weighted equally in his or her utility function, which is

$$\left(\frac{1}{2}\right) \ln(c_a) + \left(\frac{1}{2}\right) \ln(c_b),$$

where ln denotes the natural logarithm. Because the farmer lives in a climate more conducive to growing apples than bananas, however, it takes him or her one hour of work per week to grow each apple and two hours of work per week to grow each banana. If the farmer works 40 hours each week, he or she can grow any combination of $c_a$ apples and $c_b$ bananas satisfying the constraint

$$40 \geq c_a + 2c_b.$$

The farmer therefore solves the constrained maximization problem

$$\max_{c_a,c_b} \left(\frac{1}{2}\right) \ln(c_a) + \left(\frac{1}{2}\right) \ln(c_b) \text{ subject to } 40 \geq c_a + 2c_b.$$

a. As a first step in finding the farmer’s optimal choices, write down the Lagrangian for this constrained maximization problem.

b. Next, write down the first-order conditions for $c_a$ and $c_b$ that help characterize the solution to this problem.

c. Because the farmer’s utility function implies that he or she always prefers more to less, we know in advance that the optimal choices $c_a^*$ and $c_b^*$ will satisfy the binding constraint

$$40 = c_a^* + 2c_b^*$$

as well as the two first-order conditions you derived in part (b) above. Use these three equations to find the numerical values of $c_a^*$ and $c_b^*$ that solve the farmer’s problem.
2. Intertemporal Consumer Optimization

Following Irving Fisher, consider a consumer who receives income $Y_0$ in period $t = 0$ (today), which he or she divides up into an amount $c_0$ to be consumed and an amount $s$ to be saved (or borrowed, if $s < 0$), subject to the budget constraint

$$Y_0 \geq c_0 + s.$$  

Suppose that the consumer then receives income $Y_1$ in period $t = 1$ (next year), which he or she combines with his or her savings from period $t = 0$ to finance consumption $c_1$, subject to the budget constraint

$$Y_1 + (1 + r)s \geq c_1,$$

where $r$ denotes the interest rate on both saving and borrowing. As in class, we can combine these two single-period budget constraints into one present-value budget constraint

$$Y_0 + \frac{Y_1}{1 + r} \geq c_0 + \frac{c_1}{1 + r},$$

thereby also eliminating $s$ as a separate choice variable in the consumer’s problem.

Suppose, finally, that the consumer’s preferences over consumption during the two periods are described by the utility function

$$\ln(c_0) + \beta \ln(c_1),$$

where the discount factor $\beta$, satisfying $0 < \beta < 1$, measures the consumer’s patience and $\ln$ denotes the natural logarithm.

The consumer therefore solves the constrained maximization problem

$$\max_{c_0, c_1} \ln(c_0) + \beta \ln(c_1) \text{ subject to } Y_0 + \frac{Y_1}{1 + r} \geq c_0 + \frac{c_1}{1 + r}.$$  

a. Write down the Lagrangian for the consumer’s problem. Then, write down the first-order conditions for $c_0$ and $c_1$ that characterize the solution to this problem.

b. Next, assume in particular that $r = 0.10$ (10 percent) and $\beta(1 + r) = 1$ so that $1 + r = 1.1 = 11/10$ and $\beta = 10/11$. Suppose also that $Y_0 = 210$ and $Y_1 = 231$, so that

$$Y_0 + \frac{Y_1}{1 + r} = 210 + 231 \times \frac{10}{11} = 210 + 210 = 420.$$  

Use these values, together with the first-order conditions you derived in answering part (a), above, and the budget constraint, which will hold as an equality when the consumer is choosing $c_0$ and $c_1$ optimally, to find the numerical values of $c_0^*$ and $c_1^*$ that solve the consumer’s problem.

c. Use your solutions to part (b), above, to answer the question: at $t = 0$, is the consumer saving or borrowing?
3. Contingent Claims and Consumption Plans

As we discussed in class, Kenneth Arrow and Gerard Debreu worked in the late 1950s and early 1960s to extend the theory of consumer decision-making to the case of uncertainty. To do this, Arrow and Debreu imagined that consumers trade in contingent claims in order to implement state-contingent consumption plans. In the simplest version of their framework, there are two periods – this year ($t = 0$) and next year ($t = 1$) – and two possible states of the world next year – a good state that occurs with probability $\pi$ and a bad state that occurs with probability $1 - \pi$.

In this version of the Arrow-Debreu model, a contingent claim for the good state sells for $q^G$ units of consumption at $t = 0$ and pays off one unit of consumption in the good state and zero units of consumption in the bad state at $t = 1$. Likewise, a contingent claim for the bad state sells for $q^B$ units of consumption at $t = 0$ and pays off one unit of consumption in the bad state and zero units of consumption in the good state at $t = 1$. Arrow and Debreu then allowed consumers to engage in four types of trading strategies. Specifically, they could (i) buy (take a long position in) the contingent claim for the good state; (ii) sell short (take a short position in) the contingent claim for the good state; (iii) buy (take a long position in) the contingent claim for the bad state; and (iv) sell short (take a short position in) the contingent claim for the bad state.

a. Which one of the four trading strategies listed above should a consumer use if he or she wants to increase consumption at $t = 0$ and decrease consumption in the bad state at $t = 1$, without changing consumption in the good state at $t = 1$? Note: To answer this question and the next two in parts (b) and (c) below, you can just say which strategy the consumer should use, you don’t have to explain why.

b. Which one of the four trading strategies should a consumer use if he or she wants to decrease consumption at $t = 0$ and increase consumption in the bad state at $t = 1$, without changing consumption in the good state at $t = 1$?

c. Which one of the four trading strategies should a consumer use if he or she wants to decrease consumption at $t = 0$ and increase consumption in the good state at $t = 1$, without changing consumption in the bad state at $t = 1$?
4. Option Pricing

Consider another economic environment in which there are two periods, \( t = 0 \) and \( t = 1 \), and two possible states at \( t = 1 \): a good state that occurs with probability \( \pi = 1/2 \) and a bad state that occurs with probability \( 1 - \pi = 1/2 \). Suppose, initially, that two assets trade in this economy. A risky stock sells for \( q_s = 2 \) at \( t = 0 \), \( P^G = 5 \) in the good state at \( t = 1 \), and \( P^B = 2 \) in the bad state at \( t = 1 \). And a risk-free bond sells for \( q_b = 0.80 \) at \( t = 0 \) and pays off 1 in both states at \( t = 1 \).

a. Suppose now that a call option on the stock with strike price \( K_1 = 3 \) begins to trade. This call option gives the holder the right, but not the obligation, to buy one share of the stock at the price \( K_1 = 3 \) at \( t = 1 \). What will the payoffs to an owner of this call option be in the good state and bad state at \( t = 1 \)?

b. Use your answers to part (a), above, together with the information about the stock and bond given previously, to deduce the price at which the call option should trade at \( t = 0 \) if there are to be no arbitrage opportunities across the stock, bond, and options markets.

c. Suppose, finally, that another call option on the stock begins trading, this one with strike price \( K_2 = 4 \). At what price will this second call option trade at \( t = 0 \) if there are to be no arbitrage opportunities across all asset markets?
5. Pricing Risk-Free Assets

Assume, for all assets in this example, that there is no uncertainty: all payments promised by all assets get received for sure. Suppose, initially, that three discount bonds are traded. A one-year discount bond sells for \( P_1 = 0.90 \) today and pays off 1 for sure one year from now. A two-year discount bond sells for \( P_2 = 0.80 \) today and pays off 1 for sure at two years from now. And a three-year discount bond sells for \( P_3 = 0.70 \) today and pays off 1 for sure three years from now.

a. Suppose now that a new risk-free asset begins trading: a two-year coupon bond that makes an interest (coupon) payment of 10 one year from now, another interest payment of 10 two years from now, and then returns its face (or par) value of 100 two years from now. At what price \( P_{2C} \) will this coupon bond sell for today if there are to be no arbitrage opportunities across the markets for discount and coupon bonds?

b. Suppose next that another new risk-free asset begins trading: a three-year coupon bond that makes an interest (coupon) payment of 10 one year from now, another interest payment of 10 two years from now, a third interest payment of 10 three years from now, and then returns its face (or par) value of 100 three years from now. At what price \( P_{3C} \) will this coupon bond sell for today if there are to be no arbitrage opportunities across the markets for discount and coupon bonds?

c. Suppose, finally, that yet another risk-free asset begins trading, which pays off 1 for sure one year from now and 1 for sure two years from now. If the initial price of this new asset is \( P_A = 1.80 \), there will be an arbitrage opportunity across the market for this new asset and the market for discount bonds. As traders exploit this arbitrage opportunity, what will happen to the price of this new asset: will it rise or fall? Note: to answer this question, you only have to say whether the price will rise or fall, you don’t have to explain why.
Solutions to Midterm Exam

ECON 337901 - Financial Economics
Boston College, Department of Economics
Spring 2018

Tuesday, March 20, 10:30 - 11:45am

1. Apple and Banana Farming

The farmer solves the constrained maximization problem

\[
\max_{c_a, c_b} \frac{1}{2} \ln(c_a) + \frac{1}{2} \ln(c_b) \quad \text{subject to} \quad 40 \geq c_a + 2c_b.
\]

a. The Lagrangian for the farmer’s problem is

\[
L(c_a, c_b, \lambda) = \frac{1}{2} \ln(c_a) + \frac{1}{2} \ln(c_b) + \lambda (40 - c_a - 2c_b).
\]

b. The first-order conditions are

\[
\frac{1}{2c^*_a} - \lambda^* = 0
\]
for \(c_a\), and

\[
\frac{1}{2c^*_b} - \lambda^* \cdot 2 = 0
\]
for \(c_b\).

c. The two first-order conditions and the binding constraint

\[
40 = c^*_a + 2c^*_b
\]
form a system of three equations in the three unknowns: \(c^*_a\), \(c^*_b\), and \(\lambda^*\). Although there are many ways of solving this system of the equations, one is to rearrange the first-order conditions so that they become equations relating \(c^*_a\) and \(c^*_b\) to \(\lambda^*\):

\[
c^*_a = \frac{1}{2 \lambda^*}
\]
and

\[
c^*_b = \frac{1}{4 \lambda^*}.
\]

Substituting these equations into the binding constraint yields

\[
40 = \frac{1}{2 \lambda^*} + 2 \left( \frac{1}{4 \lambda^*} \right) = \frac{1}{2 \lambda^*} + \frac{1}{2 \lambda^*} + \frac{2}{2 \lambda^*} = \frac{1}{\lambda^*},
\]

which provides the numerical solution \(\lambda^* = 1/40\). The previous equations for \(c^*_a\) and \(c^*_b\) then provide the numerical solutions \(c^*_a = 20\) and \(c^*_b = 10\). Notice that these solutions imply that the farmer spends 20 hours per week growing apples and 20 hours per week growing bananas, consistent with the equal weighting of apples and bananas in his or her utility function.
The consumer solves the constrained maximization problem

\[
\max_{c_0, c_1} \ln(c_0) + \beta \ln(c_1) \text{ subject to } Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.
\]

a. The Lagrangian for the consumer’s problem is

\[
L(c_0, c_1, \lambda) = \ln(c_0) + \beta \ln(c_1) + \lambda \left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r}\right),
\]

and the first-order conditions are

\[
\frac{1}{c_0^*} - \lambda^* = 0
\]

and

\[
\frac{\beta}{c_1^*} - \lambda^* \left(\frac{1}{1+r}\right) = 0.
\]

b. To find the numerical values of \(c_0^*\) and \(c_1^*\) that solve the consumer’s problem, start by noting that when \(\beta(1+r) = 1\) as assumed, the first-order conditions imply

\[
c_0^* = \frac{1}{\lambda^*}
\]

and

\[
c_1^* = \frac{\beta(1+r)}{\lambda^*} = \frac{1}{\lambda^*}.
\]

Next, substitute these expressions for \(c_0^*\) and \(c_1^*\), together with the values \(1+r = 1.1 = 11/10\), \(Y_0 = 210\), and \(Y_1 = 231\), into the binding budget constraint to solve for \(\lambda^*\):

\[
Y_0 + \frac{Y_1}{1+r} = c_0 + \frac{c_1}{1+r}
\]

\[
210 + 231 \times \left(\frac{10}{11}\right) = \frac{1}{\lambda^*} + \frac{1}{\lambda^*} \times \left(\frac{10}{11}\right)
\]

\[
420 = \frac{1}{\lambda^*} \times \left(\frac{21}{11}\right)
\]

\[
\frac{1}{\lambda^*} = 420 \times \left(\frac{11}{21}\right)
\]

\[
\frac{1}{\lambda^*} = 220
\]

\[
\lambda^* = \frac{1}{220}.
\]

Finally, substitute this solution for \(\lambda^*\) into the previous expressions for \(c_0^*\) and \(c_1^*\) to obtain

\[
c_0^* = c_1^* = 220.
\]
c. At \( t = 0 \), the consumer’s spending \( c_0^* = 220 \) is higher than his or her income \( Y_0 = 210 \). Therefore, the consumer is borrowing.

3. Contingent Claims and Consumption Plans

In the Arrow-Debreu model with periods \( t = 0 \) and \( t = 1 \) and with good and bad states in period \( t = 1 \):

a. A consumer who wants to increase consumption at \( t = 0 \) and decrease consumption in the bad state at \( t = 1 \) without changing consumption in the good state at \( t = 1 \) should sell short the contingent claim for the bad state.

b. A consumer who wants to decrease consumption at \( t = 0 \) and increase consumption in the bad state at \( t = 1 \) without changing consumption in the good state at \( t = 1 \) should buy the contingent claim for the bad state.

c. A consumer who wants to decrease consumption at \( t = 0 \) and increase consumption in the good state at \( t = 1 \) without changing consumption in the bad state at \( t = 1 \) should buy the contingent claim for the good state.

4. Option Pricing

There are again two periods, \( t = 0 \) and \( t = 1 \), and two possible states at \( t = 1 \): a good state that occurs with probability \( \pi = 1/2 \) and a bad state that occurs with probability \( 1 - \pi = 1/2 \). Initially, two assets trade in this economy. A risky stock sells for \( q^s = 2.20 \) at \( t = 0 \), \( P^G = 5 \) in the good state at \( t = 1 \), and \( P^B = 2 \) in the bad state at \( t = 1 \). And a risk-free bond sells for \( q^b = 0.80 \) at \( t = 0 \) and pays off 1 in both states at \( t = 1 \).

a. A call option with strike price \( K_1 = 3 \) gives the holder the right, but not the obligation, to buy one share of the stock at the price \( K_1 = 3 \) at \( t = 1 \). In the good state, the holder will find it optimal to exercise the option, yielding a payoff of \( P^G - K_1 = 5 - 3 = 2 \). In the bad state, the holder will find it optimal to let the option expire, yielding a payoff of zero.

b. To deduce the price of the option, follow Robert Merton by finding the portfolio consisting of \( s \) shares of stock and \( b \) bonds that replicates its payoffs. In the good state, the option pays off 2, the stock pays off 5, and the bond pays off 1; therefore

\[
2 = 5s + b.
\]

In the bad state, the option pays off 0, the stock pays off 2, and the bond pays off 1; therefore

\[
0 = 2s + b.
\]

Eliminating \( b \) by subtracting the second equation from the first yields

\[
2 = 3s
\]
or $s = 2/3$. Substituting this solution for $s$ into either of the previous two equations yields $b = -4/3$. Evidently, the option’s payoffs are replicated by a portfolio that takes a long position in the stock and a short position in the bond. If there are to be no arbitrage opportunities across the markets for stocks, bonds, and options, the option price must equal the cost of assembling the portfolio of the stock and bond, which is

$$q^s s + q^b b = 2.20 \times (2/3) + 0.80 \times (-4/3) = 4.40/3 - 3.20/3 = 1.20/3 = 0.40.$$  

c. A call option with strike price $K_2 = 4$ provides the holder with a payoff of 1 in the good state and 0 in the bad. One could use the same procedure as in part (b) to deduce the price of this second option, but an easier way simply notes that this call option has payoffs that can be replicated by buying $1/2$ of the first option. Therefore, this second option’s price must be half of the first option’s price: 0.20.

5. Pricing Risk-Free Assets

Three risk-free discount bonds are traded. A one-year discount bond sells for $P_1 = 0.90$ today and pays off 1 for sure one year from now. A two-year discount bond sells for $P_2 = 0.80$ today and pays off 1 for sure at two years from now. And a three-year discount bond sells for $P_3 = 0.70$ today and pays off 1 for sure three years from now.

a. The cash flows from a two-year coupon bond makes an interest payment of 10 one year from now, another interest payment of 10 two years from now, and then returns its face (or par) value of 100 two years from now can be replicated by buying 10 one-year discount bonds and 110 two-year discount bonds. The absence of arbitrage opportunities requires that the price of the coupon bond equal to cost of assembling the portfolio of discount bonds. Therefore,

$$P^C_2 = 10(0.90) + 110(0.80) = 9 + 88 = 97.$$  

b. The cash flows from a three-year coupon bond that makes an interest payment of 10 each year, every year, for the next three years before returning face value of 100 three years from now can be replicated by buying 10 one-year discount bonds, 10 two-year discount bonds, and 110 three-year discount bonds. The absence of arbitrage opportunities requires that the price of the coupon bond equal to cost of assembling the portfolio of discount bonds. Therefore,

$$P^C_3 = 10(0.90) + 10(0.80) + 110(0.70) = 9 + 8 + 77 = 94.$$  

c. A risk-free asset that pays off 1 for sure one year from now and 1 for sure two years from now can be replicated by a portfolio consisting of a one-year discount bond and a two-year discount bond. If the initial price of this new asset is $P^A = 1.80$, there will be an arbitrage opportunity across markets because the portfolio of discount bonds costs only 1.70. As traders sell the new asset and buy the portfolio of discount bonds to exploit this arbitrage opportunity, the price $P^A$ of the new asset will fall.
1. Criteria for Choice Over Risky Prospects

Consider an economic environment in which there are two possible future states: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$. Three assets are traded, with percentage returns in the two states tabulated below:

<table>
<thead>
<tr>
<th>Percentage Return in the Good State ($\pi = 1/2$)</th>
<th>Percentage Return in the Bad State ($1 - \pi = 1/2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>10</td>
</tr>
<tr>
<td>Asset 2</td>
<td>20</td>
</tr>
<tr>
<td>Asset 3</td>
<td>30</td>
</tr>
</tbody>
</table>

a. Does any of the three assets display state-by-state dominance over both of the other two? If so, which one?

b. Does any of the three assets display mean-variance dominance over both of the other two? If so, which one?

c. Suppose an investor is risk averse but prefers more to less. Can you tell which one of these three assets he or she will prefer over both of the other two? If so, which one?
2. Expected Utility and Risk Aversion

Consider two investors, who both have preferences described by von Neumann-Morgenstern expected utility functions, but differ in their degree of risk aversion. The first investor – investor A – has Bernoulli utility function of the form

\[ u^A(Y) = Y^{1/2} = \sqrt{Y}, \]

where \( Y \) is the value of a monetary payoff received in any given state of the world. The second investor – investor B – has Bernoulli utility function of the form

\[ u^B(Y) = -Y^{-1/2} = -\frac{1}{\sqrt{Y}}. \]

These utility functions imply that investor A’s coefficient of relative risk aversion is constant (independent of \( Y \)) and equal to \( 1/2 \), while investor B’s coefficient of relative risk aversion is constant and equal to \( 3/2 \). In that sense, investor B is more risk averse than investor A.

Suppose that there are two possible states of the world: a good state that occurs with probability \( \pi = 1/2 \) and a bad state that occurs with probability \( 1 - \pi = 1/2 \). Then either investor’s expected utility from any asset that pays off \( p_G \) in the good state and \( p_B \) in the bad state will be

\[ U^i(p_G, p_B) = (1/2)u^i(p_G) + (1/2)u^i(p_B), \]

where \( i = A \) for investor A and \( i = B \) for investor B.

a. Suppose now that investor A is offered a choice between two risky assets. Risky asset 1 pays off 36 in the good state and 4 in the bad state. Risky asset 2 pays off 16 in the good state and 9 in the bad state. Which asset will investor A choose if he or she picks the one that provides the highest expected utility?

b. Suppose next that investor B is offered the same choice between the same two risky assets described above. Which asset will investor B choose if he or she picks the one that provides the highest expected utility?

c. Suppose finally that in addition to the two risky assets described above, each investor is also offered a risk-free asset that pays off 20 in the good state and 20 in the bad state. Which asset will investor A choose now: risky asset 1, risky asset 2, or the risk-free asset? And which asset will investor B choose: risky asset 1, risky asset 2, or the risk-free asset?
3. Expected Utility and Portfolio Allocation

Consider the portfolio allocation problem faced by an investor who has initial wealth $Y_0$. This investor allocates the amount $a$ to stocks, which provide a return of $r_G$ in a good state that occurs with probability $\pi$ and a return of $r_B$ in a bad state that occurs with probability $1 - \pi$. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides the return $r_f$ in both states. The inequalities $r_G > r_f > r_B$ imply that the good state is “good,” in the sense that stocks provide a higher return than the bond, while the bad state is “bad” in the sense that stocks provide a lower return than the bond.

Suppose that the investor’s preferences can be described by a von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the form

$$u(Y) = \ln(Y),$$

where $\ln$ denotes the natural logarithm, so that his or her portfolio allocation problem can be stated mathematically as

$$\max_a \pi \ln[(1 + r_f)Y_0 + a(r_G - r_f)] + (1 - \pi) \ln[(1 + r_f)Y_0 + a(r_B - r_f)].$$

a. Write down the first-order condition for the investor’s optimal choice $a^\ast$.

b. Suppose now that, in particular, the good state occurs with probability $\pi = 2/3$ and the bad state occurs with probability $1 - \pi = 1/3$. Suppose also that initial wealth is $Y_0 = 100$, the risk-free rate is $r_f = 0.05$ (5 percent), and stocks return $r_G = 0.55$ (a 55 percent gain) in the good state and $r_B = -0.45$ (a 45 percent loss) in the bad state. Use your first-order condition from part (a), above, to find the numerical value of $a^\ast$ for this case.

c. Suppose finally that, as in part (b) from above, $\pi = 2/3$, $1 - \pi = 1/3$, $r_f = 0.05$, $r_G = 0.55$ and $r_B = -0.45$ but that the investor’s initial wealth is $Y_0 = 200$. What is the numerical value of $a^\ast$ with this larger value of $Y_0$?
4. Portfolio Allocation and the Gains from Diversification

Consider an investor who allocates the fraction \( w_1 \) of his or her portfolio to asset 1, which has an expected return of \( \mu_1 = 3 \) and variance of its random return of \( \sigma_1^2 = 7 \), the fraction \( w_2 \) of his or her portfolio to asset 2, which has an expected return of \( \mu_2 = 1 \) and variance of its random return of \( \sigma_2^2 = 1 \), and the remaining fraction \( 1 - w_1 - w_2 \) to asset 3, which has an expected return of \( \mu_3 = 2 \) and variance of its random return of \( \sigma_3^2 = 4 \). The investor’s portfolio, therefore, has expected return

\[
\mu_p = w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 = 3w_1 + w_2 + 2(1 - w_1 - w_2)
\]

and variance of its random return of

\[
\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + (1 - w_1 - w_2)^2 \sigma_3^2 = 7w_1^2 + w_2^2 + 4(1 - w_1 - w_2)^2
\]

assuming that the correlations between the three asset returns are all zero. If the investor allocates all of his or her funds to asset 3 by choosing \( w_1 = 0 \) and \( w_2 = 0 \), the portfolio has an expected return of 2 and variance of its random return of 4. The investor hopes to choose the portfolio weights optimally, however, in order to minimize the variance, subject to the constraint that the expected return still equals 2. As we discussed in class, this problem can be solved by choosing \( w_1 \) and \( w_2 \) to maximize \(-\sigma_p^2\) subject to the constraint that \( \mu_p = \bar{\mu} = 2 \). Using the expressions from above, the Lagrangian for this problem is

\[
L(w_1, w_2, \lambda) = -7w_1^2 - w_2^2 - 4(1 - w_1 - w_2)^2 + \lambda[3w_1 + w_2 + 2(1 - w_1 - w_2) - 2].
\]

a. As a first step in solving the investor’s portfolio allocation problem, write down the first-order conditions for his or her optimal choices \( w_1^* \) and \( w_2^* \) of \( w_1 \) and \( w_2 \).

b. Next, use your two first-order conditions from part (a), above, together with the constraint

\[
3w_1^* + w_2^* + 2(1 - w_1^* - w_2^*) = 2
\]

to find the numerical values of \( w_1^* \) and \( w_2^* \) that solve the investor’s problem.

c. Finally, use your solutions from part (b), above, to calculate the variance of the return on the investor’s optimal portfolio.
5. The Capital Asset Pricing Model

Let \( \tilde{r}_A \), \( \tilde{r}_B \), and \( \tilde{r}_C \) denote random returns on three risky stocks: shares in companies A, B, and C. Likewise, let \( \tilde{r}_M \) denote the random return on the stock market as a whole. Suppose these random returns have variances and covariances as follows:

<table>
<thead>
<tr>
<th>Random Return</th>
<th>Variance</th>
<th>Covariance with ( \tilde{r}_M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company A</td>
<td>( \tilde{r}_A )</td>
<td>( \sigma^2_A = 300 )</td>
</tr>
<tr>
<td>Company B</td>
<td>( \tilde{r}_B )</td>
<td>( \sigma^2_B = 200 )</td>
</tr>
<tr>
<td>Company C</td>
<td>( \tilde{r}_C )</td>
<td>( \sigma^2_C = 400 )</td>
</tr>
<tr>
<td>Market</td>
<td>( \tilde{r}_M )</td>
<td>( \sigma^2_M = 100 )</td>
</tr>
</tbody>
</table>

Assume, in answering each of the questions below, that the market’s expected return \( E(\tilde{r}_M) \) is greater than the return \( r_f \) on risk-free assets.

a. Use the information in the table to compute the CAPM “betas” for the three individual stocks.

b. According to the CAPM, will any of the individual stocks have an expected return higher than the market’s expected return \( E(\tilde{r}_M) \)? If so, which one(s)?

c. According to the CAPM, will any of the individual stocks have an expected return lower than the return \( r_f \) on risk-free assets? If so, which one(s)?
1. Criteria for Choice Over Risky Prospects

There are two possible future states – a good state that occurs with probability \( \pi = 1/2 \) and a bad state that occurs with probability \( 1 - \pi = 1/2 \) – and three assets, with percentage returns in each state as tabulated below:

<table>
<thead>
<tr>
<th>Percentage Return in the Good State (( \pi = 1/2 ))</th>
<th>Percentage Return in the Bad State (( 1 - \pi = 1/2 ))</th>
<th>( E(\tilde{r}) )</th>
<th>( \sigma(\tilde{r}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Asset 2</td>
<td>20</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>Asset 3</td>
<td>30</td>
<td>10</td>
<td>20</td>
</tr>
</tbody>
</table>

Note that the last two columns of the table report the expected return \( E(\tilde{r}) \) and the standard deviation \( \sigma(\tilde{r}) \) of the return on each asset, calculated as

\[
E(\tilde{r}_1) = \left(\frac{1}{2}\right) \times 10 + \left(\frac{1}{2}\right) \times 10 = 10, \\
\sigma(\tilde{r}_1) = \left[\left(\frac{1}{2}\right) \times (10 - 10)^2 + \left(\frac{1}{2}\right) \times (10 - 10)^2\right]^{1/2} = 0, \\
E(\tilde{r}_2) = \left(\frac{1}{2}\right) \times 20 + \left(\frac{1}{2}\right) \times 10 = 15, \\
\sigma(\tilde{r}_2) = \left[\left(\frac{1}{2}\right) \times (20 - 15)^2 + \left(\frac{1}{2}\right) \times (10 - 15)^2\right]^{1/2} = 5, \\
E(\tilde{r}_3) = \left(\frac{1}{2}\right) \times 30 + \left(\frac{1}{2}\right) \times 10 = 20, \\
\sigma(\tilde{r}_3) = \left[\left(\frac{1}{2}\right) \times (30 - 20)^2 + \left(\frac{1}{2}\right) \times (10 - 20)^2\right]^{1/2} = 10.
\]

a. Asset 3 pays off more than both asset 1 and 2 in the good state, and pays off the same as asset 1 and 2 in the bad state. It therefore displays state-by-state dominance over both of the others.

b. Although expected returns rise going down the fourth column of the table, standard deviations rise going down column 5. Since it is not possible to get a higher expected return without also having to accept a higher standard deviation, none of the assets displays mean-variance dominance over both of the others.

c. Any investor who prefers more to less will always choose an asset that exhibits state-by-state dominance over all others. In the table, this is asset 3.
2. Expected Utility and Risk Aversion

Two investors both have preferences described by von Neumann-Morgenstern expected utility functions, but investor A’s Bernoulli utility function is

\[ u(Y) = \sqrt{Y}, \]

while investor B’s Bernoulli utility is

\[ u(Y) = -\frac{1}{\sqrt{Y}}. \]

There are two states – good and bad – which occur with equal probability 1/2.

a. Investor A’s expected utility from risky asset 1, which pays off 36 in the good state and 4 in the bad state, is

\[(1/2)\sqrt{36} + (1/2)\sqrt{4} = (1/2)6 + (1/2)2 = 3 + 1 = 4.\]

Investor A’s expected utility from risky asset 2, which pays off 16 in the good state and 9 in the bad state, is

\[(1/2)\sqrt{16} + (1/2)\sqrt{9} = (1/2)4 + (1/2)3 = 2 + 1.5 = 3.5.\]

Since 4 > 3.5, investor A will choose risky asset 1.

b. Investor B’s expected utility from risky asset 1, which pays off 36 in the good state and 4 in the bad state, is

\[-(1/2)\frac{1}{\sqrt{36}} - (1/2)\frac{1}{\sqrt{4}} = -(1/2)(1/6) - (1/2)(1/2) = -1/12 - 1/4 = -4/12 = -1/3.\]

Investor B’s expected utility from risky asset 2, which pays off 16 in the good state and 9 in the bad state, is

\[-(1/2)\frac{1}{\sqrt{16}} - (1/2)\frac{1}{\sqrt{9}} = -(1/2)(1/4) - (1/2)(1/3) = -1/8 - 1/6 = -7/24.\]

Since \(-7/24 > -1/3\), investor B will choose risky asset 2.

c. The new risk-free asset pays off 20 no matter what. It therefore offers an expected payoff that is equal to the expected payoff of 20 from risky asset 1 and higher than the expected payoff of 12.5 from risky asset 2, but without any risk. Hence, even without doing any further calculations, we know that any risk-averse investor will choose the risk-free asset over the two risky alternatives. Computing expected utility from the risk-free asset for investor A,

\[(1/2)\sqrt{20} + (1/2)\sqrt{20} = \sqrt{20} = 2\sqrt{5} = 4.47 > 4 > 3.5,\]
and for investor B,
\[-(1/2) \frac{1}{\sqrt{20}} - (1/2) \frac{1}{\sqrt{20}} = - \frac{1}{\sqrt{20}} = - \frac{1}{2\sqrt{5}} = -0.22 > -7/24 > -1/3,\]
however, confirms that both get the highest expected utility from this new risk-free asset.

3. Expected Utility and Portfolio Allocation

The investor’s portfolio allocation problem is
\[
\max_a \pi \ln[(1 + r_f)Y_0 + a(r_G - r_f)] + (1 - \pi) \ln[(1 + r_f)Y_0 + a(r_B - r_f)].
\]

a. The first-order condition for the investor’s optimal choice \(a^*\) is
\[
\frac{\pi(r_G - r_f)}{(1 + r_f)Y_0 + a^*(r_G - r_f)} + \frac{(1 - \pi)(r_B - r_f)}{(1 + r_f)Y_0 + a^*(r_B - r_f)} = 0.
\]

b. With \(\pi = 2/3, \pi = 1/3, Y_0 = 100, r_f = 0.05, r_G = 0.55,\) and \(r_B = -0.45,\) the first-order condition from part (a) specializes to
\[
\frac{(2/3)(0.50)}{105 + 0.50a^*} = \frac{(1/3)(0.50)}{105 - 0.50a^*}.
\]
Solving for \(a^*\) yields:
\[
2(105 - 0.50a^*) = 105 + 0.50a^*\\
210 - a^* = 105 + 0.50a^*\\
105 = 1.5a^*\\
a^* = 105/1.5 = 105(2/3) = 35(2) = 70.
\]

c. As we discussed in class, the logarithmic Bernoulli utility function implies that the investor’s coefficient of relative risk aversion is constant and equal to one, and with constant relative risk aversion, the optimal choice of \(a^*\) scales up and down proportionally with initial wealth \(Y_0.\) Therefore, if \(Y_0\) doubles from 100 to 200, we know in advance that \(a^*\) will also double, from 70 to 140. We can confirm this, however, by substituting \(\pi = 2/3, \pi = 1/3, Y_0 = 200, r_f = 0.05, r_G = 0.55,\) and \(r_B = -0.45\) into the first-order condition from part (a) to get
\[
\frac{(2/3)(0.50)}{210 + 0.50a^*} = \frac{(1/3)(0.50)}{210 - 0.50a^*}.
\]
Solving for \(a^*\) then yields:
\[
2(210 - 0.50a^*) = 210 + 0.50a^*\\
420 - a^* = 210 + 0.50a^*\\
210 = 1.5a^*\\
a^* = 210/1.5 = 210(2/3) = 70(2) = 140.
\]
4. Portfolio Allocation and the Gains from Diversification

An investor allocates the fraction $w_1$ of his or her portfolio to asset 1, which has an expected return of $\mu_1 = 3$ and variance of its random return of $\sigma_1^2 = 7$, the fraction $w_2$ of his or her portfolio to asset 2, which has an expected return of $\mu_2 = 1$ and variance of its random return of $\sigma_2^2 = 1$, and the remaining fraction $1 - w_1 - w_2$ to asset 3, which has an expected return of $\mu_3 = 2$ and variance of its random return of $\sigma_3^2 = 4$. The investor’s portfolio, therefore, has expected return

$$\mu_p = w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 = 3w_1 + w_2 + 2(1 - w_1 - w_2)$$

and variance of its random return of

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + (1 - w_1 - w_2)^2 \sigma_3^2 = 7w_1^2 + w_2^2 + 4(1 - w_1 - w_2)^2$$

assuming that the correlations between the three asset returns are all zero. If the investor allocates all of his or her funds to asset 3 by choosing $w_1 = 0$ and $w_2 = 0$, the portfolio has an expected return of 2 and variance of its random return of 4. The investor hopes to choose the portfolio weights optimally, however, in order to minimize the variance, subject to the constraint that the expected return still equals 2. This problem can be solved by choosing $w_1$ and $w_2$ to maximize $-\sigma_p^2$ subject to the constraint that $\mu_p = \bar{\mu} = 2$. Using the expressions from above, the Lagrangian for this problem is

$$L(w_1, w_2, \lambda) = -7w_1^2 - w_2^2 - 4(1 - w_1 - w_2)^2 + \lambda[3w_1 + w_2 + 2(1 - w_1 - w_2) - 2].$$

a. The first-order conditions can be obtained by differentiating the Lagrangian with respect to $w_1$ and $w_2$ and, in each case, setting the result equal to zero:

$$-14w_1^* + 8(1 - w_1^* - w_2^*) + \lambda^*(3 - 2) = 0$$

and

$$-2w_2^* + 8(1 - w_1^* - w_2^*) + \lambda^*(1 - 2) = 0.$$

b. Since the constraint

$$3w_1^* + w_2^* + 2(1 - w_1^* - w_2^*) = 2$$

implies that

$$w_1^* - w_2^* = 0,$$

the optimal portfolio must allocate equal shares to assets 1 and 2. Therefore, let $w^* = w_1^* = w_2^*$, and substitute this common value for $w_1^*$ and $w_2^*$ into the first-order conditions from part (a), so as to write them more simply as

$$-14w^* + 8(1 - 2w^*) + \lambda^* = 0$$

and

$$-2w^* + 8(1 - 2w^*) - \lambda^* = 0.$$
Eliminating $\lambda^*$ by adding these two equations yields

$$-16w^* + 16(1 - 2w^*) = 0$$

or

$$(16 + 32)w^* = 16$$

or

$$w^* = 1/3.$$  

Evidently, it is optimal for the investor to allocate equal 1/3 shares of his or her portfolio to each of the three assets:

$$w_1^* = w_2^* = w_3^* = 1/3.$$  

c. Substituting the optimal choices for the portfolio weights into the expression

$$\sigma_p^2 = 7w_1^2 + w_2^2 + 4(1 - w_1 - w_2)^2$$

for the variance of the portfolio’s random return shows that the optimal portfolio has

$$\sigma_p^2 + (1/3)^2(7 + 1 + 4) = 12/9 = 1.33,$$

considerably smaller than the value of 2 that the investor would get simply by allocating all of his or her funds to asset 3.

5. The Capital Asset Pricing Model

Let $\tilde{r}_A$, $\tilde{r}_B$, and $\tilde{r}_C$ denote random returns on three risky stocks: shares in companies A, B, and C. Likewise, let $\tilde{r}_M$ denote the random return on the stock market as a whole. Suppose these random returns have variances and covariances as follows:

<table>
<thead>
<tr>
<th>Random Return</th>
<th>Variance</th>
<th>Covariance with $\tilde{r}_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company A</td>
<td>$\tilde{r}_A$</td>
<td>$\sigma_A^2 = 300$</td>
</tr>
<tr>
<td>Company B</td>
<td>$\tilde{r}_B$</td>
<td>$\sigma_B^2 = 200$</td>
</tr>
<tr>
<td>Company C</td>
<td>$\tilde{r}_C$</td>
<td>$\sigma_C^2 = 400$</td>
</tr>
<tr>
<td>Market</td>
<td>$\tilde{r}_M$</td>
<td>$\sigma_M^2 = 100$</td>
</tr>
</tbody>
</table>

Assume, also, that the market’s expected return $E(\tilde{r}_M)$ is greater than the return $r_f$ on risk-free assets.
a. In general, the CAPM beta for asset $j$ is

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2},$$

where $\sigma_{jM}$ is the covariance between asset $j$’s random return $\tilde{r}_j$ and the random return on the market $\tilde{r}_M$ and $\sigma_M^2$ is the variance of the market’s random return. Therefore, using the data from the table, company A’s shares have

$$\beta_A = \frac{150}{100} = 1.5,$$

company B’s shares have

$$\beta_B = \frac{75}{100} = 0.75,$$

and company C’s shares have

$$\beta_C = \frac{-50}{100} = -0.5.$$

b. According to the CAPM, the expected return $E(\tilde{r}_j)$ on asset $j$ is determined as

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f],$$

where $E(\tilde{r}_M)$ is the expected return on the market and $r_f$ is the return on risk-free assets. Therefore, under the assumption that $E(\tilde{r}_M) > r_f$, any individual stock will have expected return higher than the market’s if its beta is greater than one and expected return lower than the market’s if its beta is less than one. Based on the betas computed in part (a), we can therefore say that according to the CAPM, only company A’s shares will have expected return higher than the market’s.

c. According to the CAPM, an individual stock will have an expected return above the return $r_f$ on risk-free assets if its beta is positive and expected return below the return on risk-free assets if its beta is negative. Based on the betas computed in part (a), we can therefore say that according to the CAPM, only company C’s shares will have expected return lower than the risk-free rate.
1. Profit Maximization

Consider a firm that hires $n$ workers in order to produce $y$ units of output according to the production function

$$y = n^{1/2}.$$ 

Let $w$ denote the wage that the firm must pay each worker in a competitive labor market. Then the firm chooses $n$ to maximize profits, defined as usual to equal revenues minus costs:

$$\max_n n^{1/2} - wn.$$ 

This is an unconstrained optimization problem, with an objective function $F(n) = n^{1/2} - wn$ that is concave, so that the first-order condition is both necessary and sufficient for the value of $n^*$ that maximizes profits.

a. Write down the first-order condition for $n^*$, using the rules of differentiation to find $F''(n^*)$ for this example.

b. Next, rearrange the first-order condition to get an equation that shows how the firm’s optimal choice $n^*$ depends on the wage rate $w$.

c. Finally, use your solution from part (b) to answer the question: when $w$ goes up, does $n^*$ rise or fall?
2. Consumer Optimization Under Uncertainty

Consider a consumer making choices under uncertainty in an environment with two periods, today \((t = 0)\) and next year \((t = 1)\), and two states, good and bad, in period \(t = 1\). Let \(c_0\) denote consumption today and \(c_{1}^{G}\) and \(c_{1}^{B}\) denote consumption in the good and bad states next year. The consumer’s expected utility is then

\[
u(c_0) + \beta \pi u(c_{1}^{G}) + \beta(1 - \pi)u(c_{1}^{B}),
\]

where \(\beta\) is a discount factor that captures the consumer’s degree of patience or impatience and \(\pi\) denotes the probability that the good state occurs next year. Similarly, let \(Y_0, Y_1^G\), and \(Y_1^B\) denote the consumer’s income today and in the good and bad states next year. The consumer’s budget constraint is then

\[
Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_{1}^{G} + q^B c_{1}^{B},
\]

where \(q^G\) is the price of a contingent claim for the good state next year and \(q^B\) is the price of a contingent claim for the bad state next year.

Now specialize this problem by setting \(\beta = \frac{1}{2}, \pi = \frac{1}{2}, 1 - \pi = \frac{1}{2}, Y_0 = 60, Y_1^G = 80, Y_1^B = 40, q^G = \frac{1}{4}, \) and \(q^B = \frac{1}{4}\). Assume, as well, that the utility function \(u(c) = \ln(c)\) takes the natural log form, so that \(u'(c) = \frac{1}{c}\). With these settings, the consumer’s problem becomes, more specifically, one of choosing \(c_0, c_{1}^{G}\), and \(c_{1}^{B}\) to maximize expected utility

\[
\ln(c_0) + \left(\frac{1}{4}\right) \ln(c_{1}^{G}) + \left(\frac{1}{4}\right) \ln(c_{1}^{B})
\]

subject to the budget constraint

\[
90 \geq c_0 + \left(\frac{1}{4}\right) c_{1}^{G} + \left(\frac{1}{4}\right) c_{1}^{B}.
\]

a. Write down the Lagrangian for the consumer’s problem.

b. Next, write down the first-order conditions for the consumer’s optimal choices \(c^{*}_0, c^{*}_{1}^{G}\), and \(c^{*}_{1}^{B}\) of \(c_0, c_{1}^{G}\), and \(c_{1}^{B}\).

c. Finally, use your first-order conditions from part (b) above, together with the binding budget constraint

\[
90 = c^{*}_0 + \left(\frac{1}{4}\right) c^{*}_{1}^{G} + \left(\frac{1}{4}\right) c^{*}_{1}^{B},
\]

to find the numerical values of \(c^{*}_0, c^{*}_{1}^{G}\), and \(c^{*}_{1}^{B}\).
3. Stocks, Bonds, and Contingent Claims

Consider an economy in which two assets are initially traded. A stock sells for $q_s = 2$ today (at $t = 0$) and pays off a large dividend $d_G = 4$ in a good state next year ($t = 1$) and a smaller dividend $d_B = 2$ in a bad state next year. A bond sells for $q_b = 0.75$ today and pays off one dollar for sure, in both the good and the bad states next year.

a. What would be the price $q^G$ today (at $t = 0$) of a contingent claim for the good state, which pays off one dollar in the good state next year (at $t = 1$) and zero in the bad state next year, if there are no arbitrage opportunities across the markets for the stock, bond, and contingent claim?

b. What would be the price $q^B$ today (at $t = 0$) of a contingent claim for the bad state, which pays off one dollar in the bad state next year (at $t = 1$) and zero in the good state next year, if there are no arbitrage opportunities across the markets for the stock, bond, and contingent claim?

c. Suppose that a new asset begins trading, which pays off $X^G = 3$ in the good state and $X^B = 1$ in the bad state next year (at $t = 1$). What would be the price $q^A$ of this asset today (at $t = 0$), if there are no arbitrage opportunities across all markets for the stock, bond, contingent claims, and this new asset?

4. Option Pricing

Consider another economy in which two assets are initially traded – note that the numbers here are slightly different from those in question 3. Here, a stock sells for $q_s = 2$ today (at $t = 0$), and sells for a high price $P^G = 5$ in a good state next year ($t = 1$) and a smaller price $P^B = 3$ in a bad state next year. A bond sells for $q_b = 0.60$ today and pays off one dollar for sure, in both the good and the bad states next year.

a. Now consider a call option, which gives the holder the right, but not the obligation, to purchase the stock at the strike price $K = 2$ next year (at $t = 1$). What are the payoffs from the option in the good state and bad state next year?

b. How many shares of stock $s$ and bonds $b$ would an investor have to buy or sell short to replicate the payoffs on this call option?

c. What would be the price $q^o$ of the option today (at $t = 0$), if there are no arbitrage opportunities across the markets for the stock, bond, and option?
5. Pricing Risk-Free Assets

Suppose that a one-year discount bond that pays off one dollar for sure one year from now sells for \( P_1 = 0.90 \) today, a two-year discount bond that pays off one dollar for sure two years from now sells for \( P_2 = 0.80 \) today, and a three-year discount bond that pays off one dollar for sure three years from now sells for \( P_3 = 0.70 \) today.

a. Consider a three-year coupon bond that makes an annual interest (coupon) payment of 100 dollars at the end of each of the next three years and also returns an additional amount (face value) of 1000 dollars at the end of the third year. What will the price of this bond be if there are no arbitrage opportunities across the markets for discount and coupon bonds?

b. Next, consider a risk-free asset that pays the holder 100 dollars for sure two years from now and 100 dollars for sure three years from now. What will the price of this asset be if there are no arbitrage opportunities across markets for all risk-free assets?

c. Finally, consider another risk-free asset that pays the holder 100 dollars for sure two years from now but then requires the holder to pay 100 dollars for sure three years from now. What will the price of this asset be if there are no arbitrage opportunities across markets for all risk-free assets?
Solutions to Midterm Exam

ECON 337901 - Financial Economics Peter Ireland
Boston College, Department of Economics Spring 2019

Tuesday, March 19, 10:30 - 11:45am

1. Profit Maximization

With the production function $y = n^{1/2}$, where $y$ is output and $n$ is the number of workers, and with $w$ denoting the real wage, the profit-maximizing firm solves

$$\max_n n^{1/2} - wn.$$

a. The first-order condition for $n^*$ is

$$(1/2)(n^*)^{-1/2} - w = 0.$$

b. To see how the firm’s optimal choice $n^*$ depends on $w$, start by rewriting the first-order condition as

$$(1/2)(n^*)^{-1/2} = w.$$

Next, multiply both sides of the equation by $(n^*)^{1/2}$ and divide both sides by $w$ to get

$$(n^*)^{1/2} = \frac{1}{2w}.$$

Finally, square both sides to get the desired solution

$$n^* = \left(\frac{1}{2w}\right)^2 = \frac{1}{4w^2}.$$

c. The solution from part (b) shows that when $w$ goes up, $n^*$ falls. This makes sense: the firm’s demand for labor is a downward-sloping function of the wage $w$.

2. Consumer Optimization Under Uncertainty

The consumer chooses $c_0$, $c_G^1$, and $c_B^1$ to maximize expected utility

$$\ln(c_0) + \left(\frac{1}{4}\right) \ln(c_G^1) + \left(\frac{1}{4}\right) \ln(c_B^1)$$

subject to the budget constraint

$$90 \geq c_0 + \left(\frac{1}{4}\right) c_G^1 + \left(\frac{1}{4}\right) c_B^1.$$
a. The Lagrangian for the consumer’s problem is
\[ L(c_0, c_1^G, c_1^B, \lambda) = \ln(c_0) + \left( \frac{1}{4} \right) \ln(c_1^G) + \left( \frac{1}{4} \right) \ln(c_1^B) + \lambda \left[ 90 - c_0 - \left( \frac{1}{4} \right) c_1^G - \left( \frac{1}{4} \right) c_1^B \right]. \]

b. The first-order conditions for the consumer’s optimal choices \( c_0^*, c_1^G^*, \) and \( c_1^B^* \) of \( c_0, c_1^G, \) and \( c_1^B \) are
\[ \frac{1}{c_0^*} - \lambda^* = 0, \]
\[ \left( \frac{1}{4} \right) \frac{1}{c_1^G^*} - \lambda^* \left( \frac{1}{4} \right) = 0, \]
and
\[ \left( \frac{1}{4} \right) \frac{1}{c_1^B^*} - \lambda^* \left( \frac{1}{4} \right) = 0. \]

c. The first-order conditions from part (b) above, imply that
\[ c_0^* = c_1^G^* = c_1^B^* = \frac{1}{\lambda^*}. \]

Substituting these conditions into the binding budget constraint yields
\[ 90 = c_0^* + \left( \frac{1}{4} \right) c_1^G^* + \left( \frac{1}{4} \right) c_1^B^* = \frac{1}{\lambda^*} \left( 1 + \frac{1}{4} + \frac{1}{4} \right) = \frac{1}{\lambda^*} \left( \frac{6}{4} \right) \]

or
\[ \frac{1}{\lambda^*} = 90 \left( \frac{4}{6} \right) = \frac{360}{6} = 60, \]

Therefore, the consumer’s optimal choices are
\[ c_0^* = c_1^G^* = c_1^B^* = 60. \]

3. Stocks, Bonds, and Contingent Claims

Two assets are initially traded. A stock sells for \( q^s = 2 \) today and pays off a large dividend \( d^G = 4 \) in a good state next year and a smaller dividend \( d^B = 2 \) in a bad state next year. A bond sells for \( q^b = 0.75 \) today and pays off one dollar for sure, in both the good and the bad states, next year.

a. To find the price of a contingent claim for the good state, consider forming a portfolio consisting of \( s \) shares of stock and \( b \) bonds that replicates the claim’s payoffs at \( t = 1 \). In the good state, the claim pays off 1 and the portfolio pays off \( 4s + b \). Therefore,
\[ 1 = 4s + b. \]
In the bad state, the claim pays off 0 and the portfolio pays off \(2s + b\). Therefore
\[
0 = 2s + b.
\]
Subtract the second of these equations from the first to get
\[
1 = 2s
\]
or \(s = 1/2\). Substitute this solution for \(s\) back into the second equation to get
\[
b = -2s
\]
or \(b = -1\). No arbitrage implies that the price of the claim must equal to the cost of assembling the portfolio. Therefore,
\[
q^G = q^s s + q^b b = (1/2)2 + 0.75(-1) = 0.25.
\]

b. Similarly, to find the price of a contingent claim for the bad state, consider forming a portfolio consisting of \(s\) shares of stock and \(b\) bonds that replicates the claim’s payoffs at \(t = 1\). In the good state, the claim pays off 0 and the portfolio pays off \(4s + b\). Therefore,
\[
0 = 4s + b.
\]
In the bad state, the claim pays off 1 and the portfolio pays off \(2s + b\). Therefore
\[
1 = 2s + b.
\]
Rewrite the first equation as
\[
b = -4s
\]
and substitute into the second to get
\[
1 = 2s + b = 2s - 4s = -2s
\]
or \(s = -1/2\). Substitute this solution for \(s\) back into the first equation to get \(b = 2\). No arbitrage implies that the price of the claim must equal to the cost of assembling the portfolio. Therefore,
\[
q^B = q^s s + q^b b = (-1/2)2 + 0.75(2) = 0.50.
\]

c. The new asset pays off \(X^G = 3\) in the good state and \(X^B = 1\) in the bad state next year (at \(t = 1\)). This asset’s payoffs next year can be replicated by a portfolio consisting of 3 claims for the good state and 1 claim for the bad state. No arbitrage implies that the price of the new asset must equal the cost of assembling the portfolio of contingent claims. Therefore,
\[
q^A = 3q^G + q^B = 3(0.25) + 0.50 = 1.25.
\]
Notice that the new asset’s payoffs next year can also be replicated by a portfolio formed by taking a long position in one share of stock \((s = 1)\) and a short position in one bond \((b = -1)\). No arbitrage also implies that the price of the new asset must equal the cost of assembling this portfolio of the stock and the bond. Therefore,
\[
q^A = q^s s + q^b b = 2 + 0.75(-1) = 1.25.
\]
Either way, the answer is the same.
4. Option Pricing

Here, a stock sells for $q^s = 2$ today (at $t = 0$), and sells for a high price $P^G = 5$ in a good state next year ($t = 1$) and a smaller price $P^B = 3$ in a bad state next year. A bond sells for $q^b = 0.60$ today and pays off one dollar for sure, in both the good and the bad states next year.

a. A call option with $K = 2$ will always be “in the money” and should therefore always be exercised at $t = 1$. It follows that, in the good state, the option’s payoff is

$$C^G = P^G - K = 5 - 2 = 3$$

and, in the bad state, the option’s payoff is

$$C^B = P^B - K = 3 - 2 = 1.$$  

b. Consider a portfolio consisting of $s$ shares of stock and $b$ bonds that replicates the payoffs on the option. In the good state, the call option pays off 3, and the portfolio pays off $5s + b$. Therefore,

$$3 = 5s + b.$$  

In the bad state, the call option pays off 1, and the portfolio pays off $3s + b$. Therefore,

$$1 = 3s + b.$$  

Subtracting the second equation from the first yields

$$2 = 2s$$

or $s = 1$. Substituting this solution for $s$ into the second equation yields

$$1 = 3s + b = 3 + b$$

or $b = -2$.

c. No arbitrage implies that the price of the option must equal to cost of assembling this portfolio of the stock and the bond. Therefore,

$$q^o = q^s s + q^b b = 2 + 0.60(-2) = 0.80.$$  

5. Pricing Risk-Free Assets

A one-year discount bond sells for $P_1 = 0.90$, a two-year discount bond sells for $P_2 = 0.80$ today, and a three-year discount bond sells for $P_3 = 0.70$ today.
a. A three-year coupon bond makes an interest payment of 100 dollars at the end of each of the next three years and also returns face value of 1000 dollars at the end of the third year. These cash flows can be replicated by buying 100 one-year discount bonds, 100 two-year discount bonds, and 1100 three-year discount bonds. No arbitrage implies that the price of the coupon bond must equal the cost of assembling the portfolio of discount bonds. Therefore, the coupon bond’s price will be

\[ P_3^C = 100(0.90) + 100(0.80) + 1100(0.70) = 90 + 80 + 770 = 940. \]

b. A risk free asset makes payments of 100 dollars two years from now and 100 dollars three years from now. These cash flows can be replicated by buying 100 two-year discount bonds and 100 three-year discount bonds. No arbitrage implies that the price of the asset must equal the cost of assembling the portfolio of discount bonds. Therefore, the asset’s price will be

\[ 100(0.80) + 100(0.70) = 80 + 70 = 150. \]

c. Another risk-free asset pays the holder 100 for sure two years from now but then requires the holder to pay 100 for sure three years from now. These cash flows can be replicated by buying 100 two-year discount bonds and selling short 100 three-year discount bonds. No arbitrage implies that the price of the asset must equal the cost of assembling the portfolio of discount bonds. Therefore, the asset’s price will be

\[ 100(0.80) - 100(0.70) = 80 - 70 = 10. \]
1. Insurance

Consider a consumer with income of 100 dollars who faces a 50 percent probability of suffering a loss that reduces his or her income to 25 dollars. Suppose that this consumer can buy an insurance policy for $x$ dollars that protects him or her fully by paying off 75 dollars if the loss occurs. Finally, assume that the consumer’s preferences can be described by a von Neumann-Morgenstern expected utility function with Bernoulli utility function of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma},$$

with $\gamma = 2$.

a. Write down an expression for the consumer’s expected utility if he or she decides to buy the insurance.

b. Write down an expression for the consumer’s expected utility if he or she decides not to buy the insurance.

c. Find the value $x^*$ of the premium $x$ that makes the investor exactly indifferent between buying and not buying the insurance.
2. Criteria for Choice Over Risky Prospects

Consider an economic environment in which there are two possible states next year: a good state that occurs with probability \( \pi = 1/2 \) and a bad state that occurs with probability \( 1 - \pi = 1/2 \). Two assets are traded: asset 1 and asset 2. For each asset, the percentage returns in each state, the expected return \( E(\tilde{R}) \), and the standard deviation of the random return \( \sigma(\tilde{R}) \) are tabulated below:

<table>
<thead>
<tr>
<th>Percentage Return in the Good State (( \pi = 1/2 ))</th>
<th>Percentage Return in the Bad State (( 1 - \pi = 1/2 ))</th>
<th>Expected Return of Return</th>
<th>Standard Deviation of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>44</td>
<td>21</td>
<td>32.5</td>
</tr>
<tr>
<td>Asset 2</td>
<td>96</td>
<td>0</td>
<td>48</td>
</tr>
</tbody>
</table>

a. Does either of the two assets display state-by-state dominance over the other? If so, which one? Does either of the two assets display mean-variance dominance over both of the other? If so, which one?

b. Assume for simplicity that the return on risk-free assets equals zero: \( r_f = 0 \). Does either of the two assets have a higher Sharpe ratio that the other? If so, which one?

d. Consider an investor who has 100 dollars to invest and must choose between asset 1 and asset 2. If he or she invests in asset 1, he or she will end up with 144 dollars in the good state and 121 dollars in the bad state next year. If he or she invests in asset 2, he or she will end up with 196 dollars in the good state and 100 dollars in the bad state next year. If this investor has preferences that can be described by an von Neumann-Morgenstern expected utility function with Bernoulli utility function of the form

\[
u(Y) = Y^{1/2} = \sqrt{Y},
\]

where \( Y \) is the amount of money he or she has in either state next year, will this investor prefer asset 1 to asset 2, prefer asset 2 to asset 1, or be indifferent between the two? Note: For this problem, it may be helpful for you to recall that \( 10^2 = 100, \ 11^2 = 121, \ 12^2 = 144, \ 13^2 = 169, \) and \( 14^2 = 196 \).
3. Wealth, Risk Aversion, and Portfolio Allocation

Consider the portfolio allocation problem faced by an investor who has initial wealth $Y_0 = 100$. This investor allocates the amount $a$ to stocks, which provide a return of $r_G = 0.16$ (16 percent) in a good state that occurs with probability $\pi = 1/2$ and a return of $r_B = 0.02$ (2 percent) in a bad state that occurs with probability $1 - \pi = 1/2$. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides a return of $r_f = 0.08$ in both states.

Suppose that the investor’s preferences can be described by a von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the form

$$u(Y) = \ln(Y),$$

where $\ln$ denotes the natural logarithm, so that his or her portfolio allocation problem can be stated mathematically as

$$\max_a \pi \ln[(1 + r_f)Y_0 + a(r_G - r_f)] + (1 - \pi) \ln[(1 + r_f)Y_0 + a(r_B - r_f)].$$

a. Using the specific values for $Y_0$, $r_G$, $r_B$, $r_f$, and $\pi$ given above, find the numerical value of the investor’s optimal choice $a^*$. 

b. Suppose instead that the investor’s initial wealth is $Y_0 = 200$, but $r_G = 0.16$, $r_B = 0.02$, $r_f = 0.08$, and $\pi = 1/2$ remain the same as above. What is the numerical value of the investor’s optimal choice $a^*$ now?

c. Finally, go back to assuming that $Y_0 = 100$, $r_G = 0.16$, $r_B = 0.02$, $r_f = 0.08$, and $\pi = 1/2$ as in part (a), but suppose that instead of having the logarithmic form, the investor’s Bernoulli utility function is

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

with $\gamma = 1/2$. Will the investor’s optimal choice $a^*$ in this case be larger than, smaller than, or the same as it was in part (a), above. Note: For this part of the problem, you don’t have to find the exact value of $a^*$, you only need to say whether it is larger than, smaller than, or the same as it was in part (a).
4. Portfolio Allocation with Mean-Variance Utility

Consider an investor with preferences over the mean $\mu_p$ and variance $\sigma_p^2$ of the return on his or her portfolio that are described by the utility function

$$U(\mu_p, \sigma_p^2) = \mu_p - \left(\frac{A}{2}\right)\sigma_p^2,$$

where a higher value of the parameter $A$ corresponds to a greater aversion to risk. Suppose that this investor allocates the fraction $w_1$ of his or her initial wealth to risky asset 1, with expected return $\mu_1 = 2$ and standard deviation of its random return equal to $\sigma_1 = 2$, the fraction $w_2$ of initial wealth to risky asset two, with expected return $\mu_2 = 5$ and standard deviation of its random return equal to $\sigma_2 = 2$, and the remaining fraction $1 - w_1 - w_2$ to a risk-free asset with return $r_f = 1$.

Assuming that the correlation between the two risk asset returns is zero, the investor’s portfolio will have expected return

$$\mu_p = (1 - w_1 - w_2)r_f + w_1\mu_1 + w_2\mu_2 = 1 - w_1 - w_2 + 2w_1 + 5w_2 = 1 + w_1 + 4w_2,$$

and variance

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 = 4w_1^2 + 4w_2^2.$$

Therefore, the investor chooses $w_1$ and $w_2$ to maximize

$$\mu_p - \left(\frac{A}{2}\right)\sigma_p^2 = 1 + w_1 + 4w_2 - 2A(w_1^2 + w_2^2).$$

a. Write down the first-order conditions for the investor’s optimal choices $w_1^*$ and $w_2^*$ of $w_1$ and $w_2$.

b. Rearrange your first-order conditions from part (a), above, to obtain solutions that show how $w_1^*$ and $w_2^*$ depend on the risk-aversion parameter $A$.

c. Use your solutions from part (b), above, to answer the question: when the investor’s degree of risk aversion as measured by the parameter $A$ increases, do the optimal portfolio weights $w_1^*$ and $w_2^*$ on the two risky assets rise or fall?
5. The Capital Asset Pricing Model

The table below reports the variances $\sigma_i^2$ of four risky asset returns: the return $\tilde{r}_M$ on the stock market as a whole, and the returns $\tilde{r}_1$, $\tilde{r}_2$, $\tilde{r}_3$ on three risky stocks. The table also reports the covariance $\sigma_{iM}$ of each risky stock return with the market return.

<table>
<thead>
<tr>
<th></th>
<th>Variance ($\sigma_i^2$)</th>
<th>Covariance ($\sigma_{iM}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market $\tilde{r}_m$</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Asset 1 $\tilde{r}_1$</td>
<td>4.00</td>
<td>1.50</td>
</tr>
<tr>
<td>Asset 2 $\tilde{r}_2$</td>
<td>9.00</td>
<td>0.75</td>
</tr>
<tr>
<td>Asset 3 $\tilde{r}_3$</td>
<td>4.00</td>
<td>$-1.50$</td>
</tr>
</tbody>
</table>

Assuming that the expected return $E(\tilde{r}_M)$ on the market is greater than the risk-free rate $r_f$, please indicate whether, according to the Capital Asset Pricing Model (CAPM), each of the statements below is true or false. Note: For this question, you only need to say whether the statement is true or false; you don’t have to explain why.

a. The expected return on asset 1 $E(\tilde{r}_1)$ should be higher than the expected return on the market $E(\tilde{r}_M)$.

b. The expected return on asset 2 $E(\tilde{r}_2)$ should be higher than the expected return on the market $E(\tilde{r}_M)$.

c. The expected return on asset 3 $E(\tilde{r}_3)$ should be lower than the risk-free rate $r_f$. 

Solutions to Final Exam

ECON 337901 - Financial Economics
Boston College, Department of Economics
Spring 2019

Thursday, May 9, 12:30 - 2:00pm

1. Insurance

A consumer with income of 100 dollars faces a 50 percent probability of suffering a loss that reduces his or her income to 25 dollars. This consumer can buy an insurance policy for $x$ dollars that protects him or her fully by paying off 75 dollars if the loss occurs. The consumer’s preferences are described by a von Neumann-Morgenstern expected utility function with Bernoulli utility function of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma},$$

with $\gamma = 2$.

a. If the consumer buys insurance, his or her expected utility is

$$u(100 - x) = \frac{(100 - x)^{1-\gamma} - 1}{1 - \gamma} = \frac{(100 - x)^{-1} - 1}{-1} = 1 - (100 - x)^{-1} = 1 - \frac{1}{100 - x}.$$  

b. If the consumer does not buy insurance, his or her expected utility is

$$(1/2)u(100 - 75) + (1/2)u(100) = \frac{1}{2} \left( \frac{25^{1-\gamma} - 1}{1 - \gamma} \right) + \frac{1}{2} \left( \frac{100^{1-\gamma} - 1}{1 - \gamma} \right)$$

$$= \frac{1}{2} \left( \frac{25^{-1} - 1}{-1} \right) + \frac{1}{2} \left( \frac{100^{-1} - 1}{-1} \right)$$

$$= \frac{1}{2} \left( 1 - \frac{1}{25} \right) + \frac{1}{2} \left( 1 - \frac{1}{100} \right)$$

$$= 1 - \frac{1}{50} - \frac{1}{200}$$

$$= 1 - \frac{5}{200}$$

$$= 1 - \frac{1}{40}$$

c. The value $x^*$ of the premium that makes the investor exactly indifferent between buying and not buying the insurance works to equate expected utility with and without the insurance. Therefore,

$$1 - \frac{1}{100 - x^*} = 1 - \frac{1}{40}$$

1
\[
\frac{1}{100 - x^*} = \frac{1}{40} \\
40 = 100 - x^* \\
x^* = 100 - 40 = 60.
\]

2. Criteria for Choice Over Risky Prospects

There are two possible states next year: a good state that occurs with probability \( \pi = 1/2 \) and a bad state that occurs with probability \( 1 - \pi = 1/2 \). Two assets are traded – asset 1 and asset 2 – with returns as tabulated below:

<table>
<thead>
<tr>
<th>Percentage Return in the Good State</th>
<th>Expected Return of Return</th>
<th>Standard Deviation of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\pi = 1/2) )</td>
<td>( (1 - \pi = 1/2) )</td>
<td>( E(\tilde{R}) )</td>
</tr>
<tr>
<td>Asset 1</td>
<td>44</td>
<td>32.5</td>
</tr>
<tr>
<td>Asset 2</td>
<td>96</td>
<td>48</td>
</tr>
</tbody>
</table>

a. Neither asset exhibits state-by-state dominance over the other: asset 2 pays off more in the good state, but asset 1 pays off more in the bad state. Nor does either asset exhibit mean-variance dominance over the other: asset 2 offers a higher expected return, but asset 1’s return has a lower standard deviation.

b. In general, the Sharpe ratio for a risky asset is defined as the expected excess return above the risk-free rate divided by the standard deviation of its random return. In the special case where the risk-free rate equals zero, however, the Sharpe ratio can be computed as \( E(\tilde{R})/\sigma(\tilde{R}) \), the expected return divided by the standard deviation. Even without doing any detailed calculations, we can see from the table that asset 1 has the higher Sharpe ratio, since its expected return is about 3 times the standard deviation whereas asset 2’s expected return equals its standard deviation.

c. If an investor with 100 chooses in asset 1, he or she will end up with 144 dollars in the good state and 121 dollars in the bad state next year. If this investor chooses asset 2, he or she will end up with 196 dollars in the good state and 100 dollars in the bad state next year. Since this investor has preferences that are described by an von Neumann-Morgenstern expected utility function with Bernoulli utility function of the form

\[
u(Y) = Y^{1/2} = \sqrt{Y},
\]

expected utility from asset 1 is

\[
(1/2)\sqrt{144} + (1/2)\sqrt{121} = (1/2)12 + (1/2)11 = 11.5
\]

and expected utility from asset 2 is

\[
(1/2)\sqrt{196} + (1/2)\sqrt{100} = (1/2)14 + (1/2)10 = 12.
\]
Comparing these numbers reveals that the investor will choose asset 2, despite the fact that asset 1 has the higher Sharpe ratio. The reason is that this investor is not very risk averse, and is willing to accept the higher risk from asset 2 in order to get the higher expected return.

### 3. Wealth, Risk Aversion, and Portfolio Allocation

An investor with initial wealth $Y_0 = 100$ allocates the amount $a$ to stocks, which provide a return of $r_G = 0.16$ (16 percent) in a good state that occurs with probability $\pi = 1/2$ and a return of $r_B = 0.02$ (2 percent) in a bad state that occurs with probability $1 - \pi = 1/2$, and the remaining amount $Y_0 - a$ to a risk-free bond, which provides a return of $r_f = 0.08$ in both states.

Since the investor’s preferences are described by a von Neumann-Morgenstern expected utility function, with logarithmic Bernoulli utility function

$$u(Y) = \ln(Y),$$

his or her portfolio allocation problem can be stated mathematically as

$$\max_a \pi \ln[(1 + r_f)Y_0 + a(r_G - r_f)] + (1 - \pi) \ln[(1 + r_f)Y_0 + a(r_B - r_f)].$$

a. Using the specific values for $Y_0$, $r_G$, $r_B$, $r_f$, and $\pi$ given earlier, the investor’s problem can be restated more specifically as

$$\max_a (1/2) \ln(108 + 0.08a) + (1/2) \ln(108 - 0.06a).$$

The first-order condition for this problem is

$$\frac{(1/2)(0.08)}{108 + 0.08a^*} - \frac{(1/2)(0.06)}{108 - 0.06a^*} = 0.$$

The first-order condition implies

$$\frac{(1/2)(0.08)}{108 + 0.08a^*} = \frac{(1/2)(0.06)}{108 - 0.06a^*}$$

$$\frac{4}{108 + 0.08a^*} = \frac{3}{108 - 0.06a^*}$$

$$4(108 - 0.06a^*) = 3(108 + 0.08a^*)$$

$$108 = [4(0.06) + 3(0.08)]a^*$$

$$108 = 0.48a^*$$

$$a^* = \frac{108}{0.48} = 225.$$
b. We know from class that, because the natural log form of the Bernoulli utility function implies that the investor’s coefficient of relative risk aversion is constant, the optimal value of $a^*$ will scale up or down proportionally with initial wealth $Y_0$. Hence, if $Y_0 = 200$, $a^* = 450$.

c. We also know from class that the constant coefficient of relative risk aversion implied by the logarithmic Bernoulli utility function equals one. If, instead, the investor’s Bernoulli utility function is

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

with $\gamma = 1/2$, his or her coefficient of relative risk aversion is constant and equal to 1/2. Since this implies that the investor is less risk averse, $a^*$ will be larger than it was in part (a).

4. Portfolio Allocation with Mean-Variance Utility

An investor with preferences over the mean $\mu_p$ and variance $\sigma_p^2$ of the return on his or her portfolio that are described by the utility function

$$U(\mu_p, \sigma_p^2) = \mu_p - \left(\frac{A}{2}\right) \sigma_p^2$$

allocates the fraction $w_1$ of his or her initial wealth to risky asset 1, with expected return $\mu_1 = 2$ and standard deviation of its random return equal to $\sigma_1 = 2$, the fraction $w_2$ of initial wealth to risky asset two, with expected return $\mu_2 = 5$ and standard deviation of its random return equal to $\sigma_2 = 2$, and the remaining fraction $1 - w_1 - w_2$ to a risk-free asset with return $r_f = 1$.

Assuming that the correlation between the two risk asset returns is zero, the investor’s portfolio will have expected return

$$\mu_p = 1 + w_1 + 4w_2,$$

and variance

$$\sigma_p^2 = 4w_1^2 + 4w_2^2.$$

Therefore, the investor chooses $w_1$ and $w_2$ to maximize

$$\mu_p - \left(\frac{A}{2}\right) \sigma_p^2 = 1 + w_1 + 4w_2 - 2A(w_1^2 + w_2^2).$$

a. The first-order conditions for the investor’s optimal choices $w_1^*$ and $w_2^*$ of $w_1$ and $w_2$ are

$$1 - 4Aw_1^* = 0$$

and

$$4 - 4Aw_2^* = 0.$$
b. The first-order conditions from part (a), above, imply that \( w_1^* \) and \( w_2^* \) are related to the risk-aversion parameter \( A \) via
\[
  w_1^* = \frac{1}{4A}
\]
and
\[
  w_2^* = \frac{1}{A}.
\]
c. The solutions from part (b), above, show that when the investor’s degree of risk aversion as measured by the parameter \( A \) increases, the optimal portfolio weights \( w_1^* \) and \( w_2^* \) on the two risky assets fall.

5. The Capital Asset Pricing Model

The table below reports the variances \( \sigma_i \) of four risky asset returns: the return \( \tilde{r}_M \) on the stock market as a whole, and the returns \( \tilde{r}_1, \tilde{r}_2, \tilde{r}_3 \) on three risky stocks. The table also reports the covariance \( \sigma_{iM} \) of each risky stock return with the market return.

<table>
<thead>
<tr>
<th></th>
<th>Variance (( \sigma_i^2 )) of Random Return</th>
<th>Covariance (( \sigma_{iM} )) with Market Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market ( \tilde{r}_m )</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Asset 1 ( \tilde{r}_1 )</td>
<td>4.00</td>
<td>1.50</td>
</tr>
<tr>
<td>Asset 2 ( \tilde{r}_2 )</td>
<td>9.00</td>
<td>0.75</td>
</tr>
<tr>
<td>Asset 3 ( \tilde{r}_3 )</td>
<td>4.00</td>
<td>-1.50</td>
</tr>
</tbody>
</table>

The Capital Asset Pricing Model relates the expected return \( E(\tilde{r}_i) \) on each risky asset \( i = 1, 2, 3 \) to the expected return \( E(\tilde{r}_M) \) on the market, the risk free rate \( r_f \), and the individual asset’s beta \( \beta_i \) according to
\[
  E(\tilde{r}_i) = r_f + \beta_i[E(\tilde{r}_M) - r_f],
\]
where the definition
\[
  \beta_i = \frac{\sigma_{iM}}{\sigma_m^2}.
\]
implies that \( \beta_1 = 1.5 \), \( \beta_2 = 0.75 \) and \( \beta_3 = -1.5 \), and where we are assuming that the expected return on the market exceeds the risk free rate, so that \( E(\tilde{r}_M) - r_f > 0 \).

a. “The expected return on asset 1 \( E(\tilde{r}_1) \) should be higher than the expected return on the market \( E(\tilde{r}_M) \).” This statement is true. Because \( \beta_1 = 1.5 \), the CAPM implies
\[
  E(\tilde{r}_1) - r_f = 1.5[E(\tilde{r}_M) - r_f] > E(\tilde{r}_M) - r_f
\]
and therefore \( E(\tilde{r}_1) > E(\tilde{r}_M) \).
b. “The expected return on asset 2 $E(\tilde{r}_2)$ should be higher than the expected return on the market $E(\tilde{r}_M)$.” This statement is false. Because $\beta_2 = 0.75$, the CAPM implies
\[ E(\tilde{r}_2) - r_f = 0.75[E(\tilde{r}_M) - r_f] < E(\tilde{r}_M) - r_f \]
and therefore $E(\tilde{r}_2) < E(\tilde{r}_M)$.

c. “The expected return on asset 3 $E(\tilde{r}_3)$ should be lower than the risk-free rate $r_f$.” This statement is true. Because $\beta_3 = -1.5$, the CAPM implies
\[ E(\tilde{r}_3) - r_f = -1.5[E(\tilde{r}_M) - r_f] < 0 \]
and therefore $E(\tilde{r}_3) < r_f$. 
This exam has five questions on six pages, including this cover sheet; before you begin, please check to make sure that your copy has all five questions and all six pages. Please note, as well, that each of the five questions has three parts. The five questions will be weighted equally in determining your overall exam score.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.
1. Unconstrained Optimization

This problem asks you to solve an unconstrained optimization problem, where the choice variable $x$ is interpreted as the fraction of total savings that a consumer takes out of a safe investment, like a bank account, and allocates to a risky investment, like the stock market, instead.

Suppose that this consumer likes to earn a higher return $R$, but dislikes volatility $V$ from his or her investments; a utility function that captures these preferences is

$$u(R,V) = R - \left(\frac{b}{2}\right) V^2.$$ 

In this utility function, $b > 0$ is a measure of the consumer’s degree of risk aversion: higher values of $b$ mean that the consumer has a stronger aversion to or distaste for risk.

Suppose, in addition, that when the consumer allocates a larger fraction $x$ of his or her savings to the stock market, the return on his or her savings increases according to

$$R = ax,$$

where $a > 0$ measures the additional return the investor earns in the stock market relative to the bank account. Suppose, as well, that increasing $x$ also increases volatility, according to

$$V = (\sqrt{s})x,$$

where $s > 0$ captures the additional volatility of the stock market.

Substituting these two expressions for $R$ and $V$ into the consumer’s utility function allows us to state his or her optimization problem as

$$\max_x ax - \left(\frac{sb}{2}\right) x^2.$$

a. Write down the first-order condition for the consumer’s optimal choice $x^*$.

b. Next, rearrange the first-order condition to get an equation that shows how the optimal choice $x^*$ depends on $a$, measuring the additional return on the stock market, $s$, measuring the riskiness of the stock market, and $b$, measuring the consumer’s aversion to risk.

c. Finally, use your solution from part (b) to answer the questions: When the additional return $a$ on stocks goes up, does $x^*$ rise or fall? When the volatility $s$ of stocks goes up, does $x^*$ rise or fall? And when risk aversion $b$ goes up, does $x^*$ rise or fall?
2. Intertemporal Consumer Optimization

Following Irving Fisher, consider a consumer who receives income $Y_0$ in period $t = 0$ (today), which he or she divides up into an amount $c_0$ to be consumed and an amount $s$ to be saved (or borrowed, if $s < 0$) subject to the budget constraint

$$Y_0 \geq c_0 + s.$$

Suppose for simplicity that the consumer receives no additional income in period $t = 1$ (next year). Hence, he or she must finance consumption $c_1$ next year solely from the return on his or her saving, subject to the budget constraint

$$(1 + r)s \geq c_1,$$

where $r$ denotes the interest rate on saving (and borrowing). As in class, we can combine these two single-period budget constraints into one present-value budget constraint

$$Y_0 \geq c_0 + \frac{c_1}{1 + r},$$

thereby eliminating $s$ as a separate choice variable.

Suppose, finally, that the consumer’s preferences over consumption during the two periods are described by the utility function

$$\ln(c_0) + \beta \ln(c_1),$$

where $\ln$ denotes the natural logarithm and $\beta$, the discount factor, measures the consumer’s patience.

The consumer therefore solves the constrained maximization problem

$$\max_{c_0, c_1} \ln(c_0) + \beta \ln(c_1) \text{ subject to } Y_0 \geq c_0 + \frac{c_1}{1 + r}.$$

a. As a first step in finding the optimal choices $c_0^*$ and $c_1^*$, write down the Lagrangian for the consumer’s problem.

b. Next, write down the two first-order conditions for the problem.

c. Finally, assume in particular that $Y_0 = 7$, $\beta = 3/4$, and $r = 1/3$. Notice that this value of $r$ implies that

$$\frac{1}{1 + r} = \frac{1}{1 + 1/3} = \frac{1}{4/3} = \frac{3}{4},$$

and therefore that $\beta(1 + r) = 1$. Use these values, together with the two first-order conditions you derived in part (b), and the consumer’s binding budget constraint

$$Y_0 = c_0^* + \frac{c_1^*}{1 + r},$$

to calculate numerical values for the optimal choices $c_0^*$ and $c_1^*$. 

3
3. Implementing State-Contingent Consumption Plans

Consider an economic environment with risk and uncertainty similar to those we analyzed many times in class. There are two periods $t = 0$ (today) and $t = 1$ (next year), and two possible states at $t = 1$: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$.

Suppose that, in this economy, a contingent claim for the good state sells for $q^G = 1/4$ units of consumption today; next year, it pays off one unit of consumption in the good state and zero in the bad state. And suppose that a contingent claim for the bad state sells for $q^B = 1/2$ units of consumption today; next year, it pays off one unit of consumption in the bad state and zero in the good state.

In each of the questions below, an “investment strategy” involves buying or selling short one or both contingent claims in order to achieve the desired rearrangement of the consumer’s consumption plans.

a. Suppose, first, that the consumer wants to increase his or her consumption in the bad state next year, and in exchange is willing to give up consumption today. Describe, briefly (a sentence or two is all it should take) an investment strategy that achieves this goal. How many units of consumption will the consumer have to give up today to get one more unit of consumption in the bad state next year?

b. Suppose, instead, that the consumer wants to increase his or her consumption today, and in exchange is willing to give up consumption in the good state next year. Describe an investment strategy that achieves this goal. How many units of consumption can the consumer obtain today by giving up one unit of consumption in the good state next year?

c. Finally, suppose that the consumer wants to increase his or her consumption in the bad state next year, and in exchange is willing to give up consumption in the good state next year; in doing so, he or she wants consumption today to remain unchanged. Describe an investment strategy that achieves these goals. How many units of consumption will the consumer have to give up in the good state next year in order to obtain one more unit of consumption in the bad state next year?
Consider another economic environment in which there are two periods, \( t = 0 \) (today) and \( t = 1 \) (next year), and two possible states \( t = 1 \): a good state that occurs with probability \( \pi = 1/2 \) and a bad state that occurs with probability \( 1 - \pi = 1/2 \).

Suppose, initially, that two assets trade in this economy. A risky stock sells for \( q^s = 3 \) at \( t = 0 \), \( P^G = 4 \) in the good state at \( t = 1 \), and \( P^B = 3 \) in the bad state at \( t = 1 \). And a risk-free bond sells for \( q^b = 0.90 \) at \( t = 0 \) and pays off 1 in both states at \( t = 1 \).

As in class, we can find portfolios of the stock and bond that replicate the payoffs on contingent claims, and thereby infer the prices at which those contingent claims should sell for at \( t = 0 \), if there are to be no arbitrage opportunities across the markets for stocks, bonds, and contingent claims. We can also do the same for stock options.

a. First, find the number of shares \( s \) and the number of bonds \( b \) that an investor would have to buy or sell short in order to replicate the payoffs from a contingent claim for the good state (that is, to construct a “synthetic” contingent claim for the good state). Then, use your answer to compute the price \( q^G \) at \( t = 0 \) of the contingent claim for the good state implied by the assumption that there are no arbitrage opportunities across the markets for the stock, bond, and contingent claim.

b. Next, find the number of shares \( s \) and the number of bonds \( b \) that an investor would have to buy or sell short in order to replicate the payoffs from a contingent claim for the bad state (that is, to construct a “synthetic” contingent claim for the bad state). Then, use your answer to compute the price \( q^B \) at \( t = 0 \) of the contingent claim for the bad state implied by the assumption that there are no arbitrage opportunities across the markets for the stock, bond, and contingent claim.

c. Finally, find the number of shares \( s \) and the number of bonds \( b \) that an investor would have to buy or sell short in order to replicate the payoff from a call option that gives the holder the right, but not the obligation, to buy a share of stock at strike price \( K = 3.50 \) at \( t = 1 \). Then, use your answer to compute the price \( q^o \) at \( t = 0 \) of the option implied by the assumption that there are no arbitrage opportunities across the markets for the stock, bond, and option.
5. Pricing Risk-Free Assets

Suppose that a one-year discount bond that pays off one dollar for sure one year from now sells for $P_1 = 0.95$ today, and a two-year discount bond that pays off one dollar for sure two years from now sells for $P_2 = 0.90$ today.

a. Consider a two-year coupon bond that makes annual interest (coupon) payments of 100 dollars at the end of each of the next two years plus a larger payment of face value of 1000 dollars at the end of the second year. In total, therefore, this coupon bond pays off 100 dollars for sure one year from now and 1100 dollars for sure two years from now. What will the price of this coupon bond be if there are no arbitrage opportunities across the markets for discount bonds and coupon bonds?

b. Suppose another risk-free asset makes annual payments of 100 dollars at the end of each of the next three years (that is, 100 dollars for sure one year from now, 100 dollars for sure two years from now, and 100 dollars for sure three years from now) and sells for 270 dollars today. Use this information, together with the information on the prices and payoffs of the one and two-year discount bonds given above, to find the price $P_3$ today of a three-year discount bond that pays off one dollar for sure three years from now, assuming again that there are no arbitrage opportunities across markets for risk-free assets.

c. Finally, consider a three-year coupon bond that makes annual interest (coupon) payments of 100 dollars at the end of each of the next three years plus a larger payment of face value of 1000 dollars at the end of the third year. In total, therefore, this coupon bond pays off 100 dollars for sure one year from now, 100 dollars for sure two years from now, and 1100 dollars for sure three years from now. What will the price of this coupon bond be if there are no arbitrage opportunities across the markets for risk-free assets?
1. Unconstrained Optimization

The consumer’s unconstrained optimization problem is

$$\max_{x} \ ax - \left( \frac{sb}{2} \right)x^2.$$ 

a. Differentiate the objective function by the choice variable $x$ to derive the first-order condition for the consumer’s optimal choice $x^*$:

$$a - (sb)x^* = 0,$$

b. Rearrange the first-order condition to get

$$x^* = \frac{a}{sb},$$

which shows how the optimal choice $x^*$ depends on $a$, measuring the additional return on the stock market, $s$, measuring the riskiness of the stock market, and $b$, measuring the consumer’s aversion to risk.

c. The solution from part (b) implies that the optimal choice $x^*$, measuring the fraction of savings allocated to the stock market, rises when the additional return $a$ on stocks goes up and falls when either the volatility $s$ of stocks goes up or when the consumer’s degree of risk aversion $b$ goes up.

2. Intertemporal Consumer Optimization

The consumer solves the constrained maximization problem

$$\max_{c_0, c_1} \ln(c_0) + \beta \ln(c_1) \text{ subject to } Y_0 \geq c_0 + \frac{c_1}{1 + r}.$$ 

a. The Lagrangian for the consumer’s problem is

$$L(c_0, c_1, \lambda) = \ln(c_0) + \beta \ln(c_1) + \lambda \left( Y_0 - c_0 - \frac{c_1}{1 + r} \right).$$
b. The two first-order conditions for the problem are
\[ \frac{1}{c_0^*} - \lambda^* = 0 \]
and
\[ \frac{\beta}{c_1^*} - \lambda^* \left( \frac{1}{1+r} \right) = 0. \]

c. When \( Y_0 = 7, \beta = 3/4, \) and \( r = 1/3, \) the first order conditions from part (b) imply that
\[ c_0^* = c_1^* = \frac{1}{\lambda^*}. \]
Evidently, it is optimal for the consumption to be constant over time. Substituting the common value \( 1/\lambda^* \) for \( c_0^* \) and \( c_1^* \) into the binding constraint
\[ 7 = c_0^* + (3/4)c_1^* \]
yields
\[ 7 = \frac{1}{\lambda^*} \left( 1 + \frac{3}{4} \right) = \frac{1}{\lambda^*} \left( \frac{7}{4} \right), \]
implying that
\[ \frac{1}{\lambda^*} = 4 \]
and therefore
\[ c_0^* = c_1^* = 4. \]

3. Implementing State-Contingent Consumption Plans

In this economy, a contingent claim for the good state sells for \( q^G = 1/4 \) units of consumption today and a contingent claim for the bad state sells for \( q^B = 1/2 \) units of consumption today.

a. If the consumer wants to increase his or her consumption in the bad state next year, and in exchange is willing to give up consumption today, he or she should buy contingent claims for the bad state next year. In particular, to get one unit of consumption in the bad state next year, he or she should buy one contingent claim for the bad state next year. Since the price of the contingent claim for the bad state is \( q^B = 1/2, \) the consumer will have to give up 1/2 unit of consumption today to get one more unit of consumption in the bad state next year.

b. If the consumer wants to increase his or her consumption today, and in exchange is willing to give up consumption in the good state next year, he or she should sell short contingent claims for the good state next year. In particular, by selling short one contingent claim for the good state, the consumer will obtain \( q^G = 1/4 \) units of consumption today, but will have to give up one unit of consumption in the good state next year.
c. If the consumer wants to increase his or her consumption in the bad state next year, and in exchange is willing to give up consumption in the good state next year, leaving consumption today unchanged, he or she should buy contingent claims for the bad state and sell short contingent claims for the good state. In particular, to obtain one more unit of consumption in the bad state next year, the consumer should buy one contingent claim for the bad state. To offset the cost \( q^B = 1/2 \) of buying the claim for the bad state, the consumer should sell short 2 contingent claims for the good state. This short sale provides the consumer with \( 2q^G = 1/2 \) today, but requires him or her to pay 2 units of consumption in the good state next year. Hence, the consumer will have to give up two units of consumption in the good state next year in order to obtain one more unit of consumption in the bad state next year.

4. Pricing Contingent Claims and Options

Initially, two assets trade in this economy. A risky stock sells for \( q^s = 3 \) at \( t = 0 \), \( P^G = 4 \) in the good state at \( t = 1 \), and \( P^B = 3 \) in the bad state at \( t = 1 \). And a risk-free bond sells for \( q^b = 0.90 \) at \( t = 0 \) and pays off 1 in both states at \( t = 1 \).

a. A contingent claim for the good state pays off one in the good state and zero in the bad state at \( t = 1 \). To replicate these payoffs, a portfolio consisting of \( s \) shares of stock and \( b \) bonds must have

\[
1 = P^G s + b = 4s + b
\]

and

\[
0 = P^B s + b = 3s + b.
\]

Subtracting the second equation from the first shows that \( s = 1 \). Substituting this solution for \( s \) back into either of the two equations reveals that \( b = -3 \). If there are no arbitrage opportunities across markets, the price of the contingent claim must equal the cost of assembling this portfolio of the stock and the bond:

\[
q^G = q^s s + q^b b = 3(1) + 0.90(-3) = 3.00 - 2.70 = 0.30.
\]

b. A contingent claim for the bad state pays off zero in the good state and one in the bad state at \( t = 1 \). To replicate these payoffs, a portfolio consisting of \( s \) shares of stock and \( b \) bonds must have

\[
0 = P^G s + b = 4s + b
\]

and

\[
1 = P^B s + b = 3s + b.
\]

Subtracting the second equation from the first shows that \( s = -1 \). Substituting this solution for \( s \) back into either of the two equations reveals that \( b = 4 \). If there are no arbitrage opportunities across markets, the price of the contingent claim must equal the cost of assembling this portfolio of the stock and the bond:

\[
q^B = q^s s + q^b b = 3(-1) + 0.90(4) = -3.00 + 3.60 = 0.60.
\]
c. A call option on the stock with strike price $K = 3.50$ will be in the money in the good state and out of the money in the bad state at $t = 1$. Therefore, its payoff in the good state is $P^G - K = 4 - 3.50 = 0.50$ and its payoff in the bad state is 0. To replicate these payoffs, a portfolio consisting of $s$ shares of stock and $b$ bonds must have

$$0.5 = P^G s + b = 4s + b$$

and

$$0 = P^B s + b = 3s + b.$$ 

Subtracting the second equation from the first shows that $s = 0.5 = 1/2$. Substituting this solution for $s$ back into either of the two equations reveals that $b = -3/2$. If there are no arbitrage opportunities across markets, the price of the call option must equal the cost of assembling this portfolio of the stock and the bond:

$$q^c = q^s s + q^b b = 3(1/2) + 0.90(-3/2) = 1.50 - 1.35 = 0.15.$$ 

We can check this answer by observing that the payoffs from the call option can also be replicated by buying $1/2$ contingent claim for the good state at cost $q^G/2 = 0.15$.

5. Pricing Risk-Free Assets

A one-year discount bond that pays off one dollar for sure one year from now sells for $P_1 = 0.95$ today, and a two-year discount bond that pays off one dollar for sure two years from now sells for $P_2 = 0.90$ today.

a. A two-year coupon bond makes annual interest payments of 100 dollars at the end of each of the next two years plus a larger payment of face value of 1000 dollars at the end of the second year. These payoffs can be replicated by buying 100 one-year discount bonds and 1100 two-year discount bonds. If there are no arbitrage opportunities across markets, the price of the coupon bond must equal the cost of assembling the portfolio of discount bonds:

$$P_C^2 = 100P_1 + 1100P_2 = 95 + 990 = 1085.$$ 

b. Another risk-free asset makes annual payments of 100 dollars at the end of each of the next three years and sells for 270 dollars today. The payoffs from this asset can be replicated by buying 100 one-year discount bonds, 100 two-year discount bonds, and 100 three-year discount bonds. If there are no arbitrage opportunities across markets, the price of the new risk-free asset must equal to cost of assembling the portfolio of discount bonds:

$$P_A = 100P_1 + 100P_2 + 100P_3,$$

where $P_3$ is the price of a three-year discount bond that pays off one dollar for sure three years from now. We know that $P_A = 270$, $P_1 = 0.95$ and $P_2 = 0.90$. Substituting these numbers into the no-arbitrage condition yields

$$270 = 95 + 90 + 100P_3.$$
which can be used to find the implied price today of the three-year discount bond:

\[ P_3 = \frac{270 - 95 - 90}{100} = \frac{85}{100} = 0.85. \]

c. A three-year coupon bond makes annual interest payments of 100 dollars at the end of each of the next three years, and a larger payment of face value of 1000 dollars at the end of the third year. These payoffs can be replicated by buying 100 one-year discount bonds and 100 two-year discount bonds, and 1100 three-year discount bonds. If there are no arbitrage opportunities across markets, the price of the coupon bond must equal the cost of assembling the portfolio of discount bonds:

\[ P_{3}^{C} = 100P_1 + 100P_2 + 1100P_3 = 95 + 90 + 935 = 1120. \]
1. Criteria for Choice Over Risky Prospects

The table below shows the percentage returns on two risky assets, asset 1 and asset 2, in an economic environment in which there are two future states: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$. The table also reports the expected return $E(\tilde{R})$ and the standard deviation of the random return $\sigma(\tilde{R})$ on the two assets.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Percentage Return in</th>
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<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Good State</td>
<td>Bad State</td>
<td>$E(\tilde{R})$</td>
<td>$\sigma(\tilde{R})$</td>
<td></td>
</tr>
<tr>
<td>Asset 1</td>
<td>8</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Asset 2</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

a. Does either asset display state-by-state dominance over the other? If so, which one?

b. Does either asset display mean-variance dominance over the other? If so, which one?

c. Assuming the risk-free interest rate is zero, does either asset have a Sharpe ratio that is larger than the other? If so, which one?
2. Expected Utility and Risk Aversion

Consider an investor whose preferences over simple lotteries can be described by the von Neumann-Morgenstern expected utility function

\[ U(x, y, \pi) = \pi u(x) + (1 - \pi) u(y), \]

where \(x\) is the “first prize,” \(y\) is the “consolation prize,” and \(\pi\) is the probability of winning the first prize. Suppose that this investor’s Bernoulli utility function \(u\), describing preferences over payoffs in any given state, takes the general form

\[ u(p) = \frac{p^{1-\gamma}}{1-\gamma}, \]

with the specific value \(\gamma = 1/2\). Then, in particular, the investor’s expected utility function becomes

\[ U(x, y, \pi) = 2\pi \sqrt{x} + 2(1 - \pi) \sqrt{y}. \]

a. Suppose this investor is given the choice between two lotteries. Lottery 1 has \((x, y, \pi) = (4, 0, 1)\), and is equivalent to getting 4 dollars for sure. Lottery 2 has \((x, y, \pi) = (6, 2, 1/2)\), and is equivalent to a bet on a coin flip that yields 6 dollars or 2 dollars with equal probability. Will the investor prefer lottery 1 to lottery 2, prefer lottery 2 to lottery 1, or be indifferent between lottery 1 and 2. *Hint:* to answer this question, you don’t actually have to compute the expected utilities from the two lotteries.

b. Suppose that the investor can still choose lottery 1, with \((x, y, \pi) = (4, 0, 1)\) (4 dollars for sure), but that lottery 2 is replaced by lottery 3, with \((x, y, \pi) = (9, 1, 1/2)\), which is equivalent to a bet on a coin flip that yields 9 dollars or 1 dollar with equal probability. Will the investor prefer lottery 1 to lottery 3, prefer lottery 3 to lottery 1, or be indifferent between lottery 1 and 3?

c. Now consider another investor, whose preferences are described by the Bernoulli utility function

\[ v(p) = \sqrt{p}, \]

and therefore has expected utility function

\[ V(x, y, \pi) = \pi v(x) + (1 - \pi) v(y) = \pi \sqrt{x} + (1 - \pi) \sqrt{y}. \]

Will this second investor prefer lottery 1 to lottery 3, prefer lottery 3 to lottery 1, or be indifferent between lottery 1 and 3?
3. The Gains from Diversification

Suppose that two risky assets are traded. Asset 1 has random return with expected value \( \mu_1 = 8 \), standard deviation \( \sigma_1 = 2 \), and variance \( \sigma_1^2 = 4 \). Asset 2 has random return with expected value \( \mu_2 = 4 \), standard deviation \( \sigma_2 = 2 \), and variance \( \sigma_2^2 = 4 \).

a. Assume first that the two asset returns are uncorrelated, with correlation \( \rho_{12} = 0 \). Calculate the expected return and the variance of the random return on a portfolio that allocates equal shares of \( w = 1/2 \) of initial wealth to asset 1 and \( 1 - w = 1/2 \) of initial wealth to asset 2.

b. Assume instead that the two asset returns are perfectly positively correlated, with correlation \( \rho_{12} = 1 \). Recalculate the variance of the random return on the equally-weighted portfolio that allocates share \( w = 1/2 \) of initial wealth to asset 1 and share \( 1 - w = 1/2 \) of initial wealth to asset 2.

c. Suppose, finally, that the two asset returns are perfectly negative correlated, with correlation \( \rho_{12} = -1 \). Recalculate the variance of the random return on the equally-weighted portfolio that allocates share \( w = 1/2 \) of initial wealth to asset 1 and share \( 1 - w = 1/2 \) of initial wealth to asset 2.
4. The Minimum Variance Frontier

Extending the example from question 3, suppose that three risky assets are traded. As before, asset 1 has random return with expected value $\mu_1 = 8$, standard deviation $\sigma_1 = 2$, and variance $\sigma_1^2 = 4$. And asset 2 has random return with expected value $\mu_2 = 4$, standard deviation $\sigma_2 = 2$, and variance $\sigma_2^2 = 4$. But now a new asset, asset 3, has random return with expected value $\mu_3 = 6$, standard deviation $\sigma_3 = 2$, and variance $\sigma_3^2 = 4$. For simplicity, assume once again that the three asset returns are uncorrelated, with $\rho_{12} = \rho_{13} = \rho_{23} = 0$.

Suppose now that a portfolio manager is assigned the task of forming a portfolio that achieves a six percent target $\bar{\mu} = 6$ for its expected return, while minimizing the variance of the portfolio’s random return. If this portfolio manager allocates the share $w_1$ of his or her total funds to asset 1, share $w_2$ of total funds to asset 2, and the remaining share $1 - w_1 - w_2$ to asset 3, the expected return on the portfolio will be

$$\mu_p = 8w_1 + 4w_2 + 6(1 - w_1 - w_2)$$

and the variance of the random return on the portfolio will be

$$\sigma_p^2 = 4w_1^2 + 4w_2^2 + 4(1 - w_1 - w_2)^2.$$ 

Clearly, the portfolio manager can hit the six-percent expected return target simply by allocating all of the funds to asset 3; he or she would then have to accept that the variance $\sigma_3^2 = 4$ of asset 3’s return would also become the variance of his or her portfolio’s random return. The question is how much better he or she can do by choosing the portfolio weights optimally instead. Recall from class that these optimal weights can be found by maximizing $-\sigma_p^2$, minus one times the variance of the portfolio’s random return, subject to the constraint that $\mu_p = \bar{\mu} = 6$. This problem can be stated mathematically as

$$\max_{w_1, w_2} -4w_1^2 - 4w_2^2 - 4(1 - w_1 - w_2)^2 \text{ subject to } 8w_1 + 4w_2 + 6(1 - w_1 - w_2) = 6.$$ 

a. Use the Lagrangian for the portfolio manager’s problem to derive the first-order conditions for the optimal choices $w_1^*$ and $w_2^*$.

b. Next, use your first-order conditions from part (a), together with the constraint

$$8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6,$$

to find the numerical values of $w_1^*$ and $w_2^*$.

c. Finally, use your results from part (b) to calculate the minimized variance $\sigma_p^{2*}$ achieved when the portfolio’s weights are chosen optimally.
5. The Capital Asset Pricing Model

Suppose that, between today and next year, the expected return on the market is $E(\tilde{r}_M) = 0.08$ (8 percent) and the risk-free rate is $r_f = 0.02$ (2 percent), and consider three individual risky assets, each of which makes a single random (risky) payoff next year. Risky asset 1 sells for price $P_1$ today and has random payoff $\tilde{C}_1$ next year; risky asset 2 sells for price $P_2$ today and has random payoff $\tilde{C}_2$ next year; and risky asset 3 sells for price $P_3$ today and has random (risky) payoff $\tilde{C}_3$ next year.

Suppose that all three assets have the same expected payoff next year; for simplicity, assume this common expected payoff equals one:

$$E(\tilde{C}_1) = E(\tilde{C}_2) = E(\tilde{C}_3) = 1.$$

Then, as we did in class, we can compute each asset’s random return

$$\tilde{r}_1 = \frac{\tilde{C}_1 - P_1}{P_1} = \frac{\tilde{C}_1}{P_1} - 1$$

$$\tilde{r}_2 = \frac{\tilde{C}_2 - P_2}{P_2} = \frac{\tilde{C}_2}{P_2} - 1$$

$$\tilde{r}_3 = \frac{\tilde{C}_3 - P_3}{P_3} = \frac{\tilde{C}_3}{P_3} - 1$$

and expected return

$$E(\tilde{r}_1) = \frac{E(\tilde{C}_1) - P_1}{P_1} = \frac{1}{P_1} - 1$$

$$E(\tilde{r}_2) = \frac{E(\tilde{C}_2) - P_2}{P_2} = \frac{1}{P_2} - 1$$

$$E(\tilde{r}_3) = \frac{E(\tilde{C}_3) - P_3}{P_3} = \frac{1}{P_3} - 1.$$

Finally, assume that asset 1’s random return $\tilde{r}_1$ is positively correlated with the market’s return $\tilde{r}_M$, asset 2’s random return $\tilde{r}_2$ is uncorrelated with the market’s return, and asset 3’s random return $\tilde{r}_3$ is negatively correlated with the market’s return.

a. According to the capital asset pricing model, which asset will have the highest expected return: asset 1, asset 2, or asset 3?

b. According to the capital asset pricing model, which asset will have the highest price: asset 1, asset 2, or asset 3?

c. According to the capital asset pricing model, will the expected return $E(\tilde{r}_2)$ on asset 2 be greater than less than, or equal to the risk-free rate $r_f$?
1. Criteria for Choice Over Risky Prospects

The two risky assets have returns described in the table below:

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<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

a. Asset 1 displays state-by-state dominance over asset 2: it provides a higher return in the good state and the same return in the bad state.

b. Neither asset displays mean-variance dominance over the other: asset 1 provides a higher expected return, but the standard deviation of its random return is larger. As we discussed in class, this is a case where the variance or standard deviation of asset 1’s return does not distinguish adequately between upside potential and downside risk: asset 1’s return is more volatile, but only because it is higher in the good state.

c. Assuming the risk-free interest rate is zero, asset 1 has Sharpe ratio $E(\tilde{R})/\sigma(\tilde{R})$ equal to $5/3 = 1.67$ and asset 2 has Sharpe ratio equal to $4/2 = 2$. Therefore, asset 2 has the higher Sharpe ratio. Again, this highlights the problem with using the standard deviation as a measure of risk: asset 2 has the higher Sharpe ratio because its return in smaller in the good state.

2. Expected Utility and Risk Aversion

An investor’s preferences over lotteries are described by the von Neumann-Morgenstern expected utility

$$U(x, y, \pi) = 2\pi \sqrt{x} + 2(1 - \pi) \sqrt{y}.$$  

a. Lottery 1 has $(x, y, \pi) = (4, 0, 1)$, and is equivalent to getting 4 dollars for sure. Lottery 2 has $(x, y, \pi) = (6, 2, 1/2)$, and is equivalent to a bet on a coin flip that yields 6 dollars or 2 dollars with equal probability. Both lotteries have expected payoff equal to 4, but lottery 1 provides 4 for sure while lottery 2 requires the investor to take the risk on a coin-flip. No risk-averse investor will take the risk. Hence, we know even without computing expected utility that the investor will prefer lottery 1.
b. Lottery 1, with \((x, y, \pi) = (4, 0, 1)\) (4 dollars for sure), provides expected utility

\[ U(4, 0, 1) = 2\sqrt{4} = 4, \]

while lottery 3, with \((x, y, \pi) = (9, 1, 1/2)\), provides expected utility

\[ U(9, 1, 1/2) = 2(1/2)\sqrt{9} + 2(1/2)\sqrt{1} = 3 + 1 = 4. \]

Therefore, the investor is indifferent between lottery 1 and lottery 3.

c. Now consider another investor, whose preferences are described by the Bernoulli utility function

\[ v(p) = \sqrt{p}, \]

and therefore has expected utility function

\[ V(x, y, \pi) = \pi v(x) + (1 - \pi)v(y) = \pi \sqrt{x} + (1 - \pi)\sqrt{y}. \]

This investor’s preferences are the same as those of the investor considered in parts (a) and (b). Even without computing expected utilities, therefore, we know that this investor will also be indifferent between lottery 1 and lottery 3.

3. The Gains from Diversification

Two risky assets are traded. Asset 1 has random return with expected value \(\mu_1 = 8\), standard deviation \(\sigma_1 = 2\), and variance \(\sigma_1^2 = 4\). Asset 2 has random return with expected value \(\mu_2 = 4\), standard deviation \(\sigma_2 = 2\), and variance \(\sigma_2^2 = 4\).

a. A portfolio that allocates the equal shares \(w = 1/2\) and \(1 - w = 1/2\) to assets 1 and 2 has expected return

\[ \mu_p = (1/2)\mu_1 + (1/2)\mu_2 = (1/2)8 + (1/2)4 = 6. \]

This expected return does not depend on the correlation \(\rho_{12}\) between the two random returns. In general, the variance of the random return on this equally-weighted portfolio is

\[
\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12} \\
= (1/2)^2\sigma_1^2 + (1/2)^2\sigma_2^2 + 2(1/2)(1/2)\sigma_1\sigma_2\rho_{12} \\
= (1/4)4 + (1/4)4 + 2(1/4)(2)(2)\rho_{12} \\
= 2 + 2\rho_{12}.
\]

Therefore, if \(\rho_{12} = 0\), \(\sigma_p^2 = 2\).

b. If, instead, \(\rho = 1\), then \(\sigma_p^2 = 4\).

c. And if \(\rho = -1\), then \(\sigma_p^2 = 0\).
4. The Minimum Variance Frontier

The portfolio manager solves

\[
\max_{w_1, w_2} -4w_1^2 - 4w_2^2 - 4(1 - w_1 - w_2)^2 \text{ subject to } 8w_1 + 4w_2 + 6(1 - w_1 - w_2) = 6.
\]

a. The Lagrangian for the portfolio manager’s problem is

\[
L(w_1, w_2, \lambda) = -4w_1^2 - 4w_2^2 - 4(1 - w_1 - w_2)^2 + \lambda[8w_1 + 4w_2 + 6(1 - w_1 - w_2) - 6].
\]

The first-order conditions for optimal weights \(w_1^*\) and \(w_2^*\) are

\[
-8w_1^* + 8(1 - w_1^* - w_2^*) + \lambda^*(8 - 6) = 0
\]

and

\[
-8w_2^* + 8(1 - w_1^* - w_2^*) + \lambda^*(4 - 6) = 0.
\]

b. The constraint

\[
8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6,
\]

implies that

\[
8w_1^* + 4w_2^* + 6 - 6w_1^* - 6w_2^* = 6
\]

or, more simply,

\[
2w_1^* - 2w_2^* = 0
\]

or, even more simply,

\[
w_1^* = w_2^*.
\]

Let \(w^*\) be the common value of \(w_1^*\) and \(w_2^*\). Then the first-order conditions imply that

\[
-8w^* + 8(1 - 2w^*) + 2\lambda^* = 0
\]

and

\[
-8w^* + 8(1 - 2w^*) - 2\lambda^* = 0.
\]

Adding these two equations to eliminate \(\lambda^*\) yields

\[
-16w^* + 16(1 - 2w^*) = 0
\]

or

\[
w^* = \frac{16}{16 + 32} = \frac{1}{3}.
\]

Evidently, it is optimal for the portfolio manager to select equal weights on all three assets, with

\[
w_1^* = w_2^* = 1 - w_1^* - w_2^* = 1/3.
\]

c. The minimized variance is therefore

\[
\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + (1 - w_1^* - w_2^*)^2 \sigma_3^2 = (1/9)4 + (1/9)4 + (1/9)4 = 3(1/9)4 = 4/3.
\]
5. The Capital Asset Pricing Model

Between today and next year, the expected return on the market is $E(\tilde{r}_M) = 0.08$ (8 percent) and the risk-free rate is $r_f = 0.02$ (2 percent). Risky asset 1 sells for price $P_1$ today and has random payoff $\tilde{C}_1$ next year; risky asset 2 sells for price $P_2$ today and has random payoff $\tilde{C}_2$ next year; and risky asset 3 sells for price $P_3$ today and has random (risky) payoff $\tilde{C}_3$ next year. All three assets have the same expected payoff, equal to one, next year:

$$E(\tilde{C}_1) = E(\tilde{C}_2) = E(\tilde{C}_3) = 1.$$  

Asset 1’s random return $\tilde{r}_1$ is positive correlated with the market’s return $\tilde{r}_M$, asset 2’s random return $\tilde{r}_2$ is uncorrelated with the market’s return, and asset 3’s random return $\tilde{r}_3$ is negatively correlated with the market’s return.

a. According to the capital asset pricing model, the expected return on each asset $j = 1, 2, 3$ is determined as

$$E(\tilde{r}_j) = r_f + \beta_j [E(\tilde{r}_M) - r_f],$$

where the CAPM $\beta_j$ for each asset is the covariance between asset $j$’s random return and the return on the market divided by the variance of the return on the market:

$$\beta_j = \frac{\text{cov}(\tilde{r}_j, \tilde{r}_M)}{\text{var}(\tilde{r}_M)}.$$  

Thus, asset 1 has a positive beta, asset 2 has a zero beta, and asset 3 has a negative beta. Since $E(\tilde{r}_M) - r_f > 0$, it follows from these observations that asset 1 will have the highest expected return.

b. From the same observations made in part (a), it also follows that asset 3 will have the lowest expected return. Because all three assets have the same expected payoff next year, asset 3 must provide this lowest expected return by selling for the highest price.

c. Since asset 2 has $\beta_j = 0$, the CAPM implies that its expected return $E(\tilde{r}_2)$ will equal the risk-free rate $r_f$. 
