Midterm Exam

ECON 337901 - Financial Economics — Peter Ireland
Boston College, Department of Economics — Spring 2025

Due Tuesday, March 25

This exam has five questions on 12 pages; before you begin, please check to make sure that your copy has all five questions and all 12 pages. Please note, as well, that question 1 has three parts, question 2 has two parts, question 3 has one part, question 4 has three parts, and question 4 has one part. Each part of each question will be weighted equally in determining your overall exam score, so that question 1 is worth 30 points, question 2 is worth 20 points, question 3 is worth 10 points, question 4 is worth 30 points, and question 5 is worth 10 points, for a total of 100 points overall.

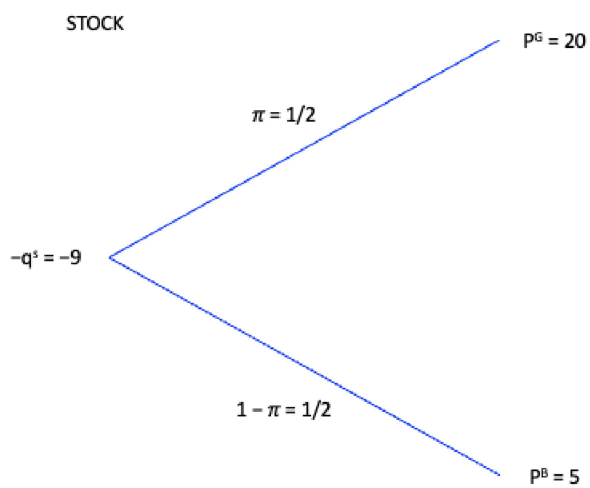
Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

This is an open-book exam, meaning that it is fine for you to consult your notes, material from the course and Canvas webpages, and other printed and electronic resources when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Futures Pricing

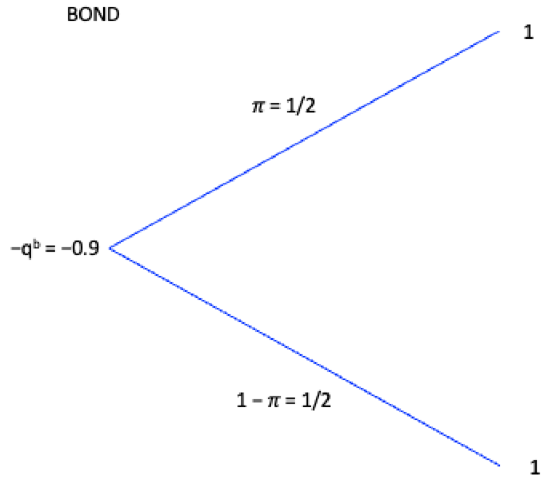
As discussed in the introductory video for this class, a *futures contract* is similar to a stock option, except that it *obligates* the buyer (an investor taking a long position) to buy a share of stock at the "delivery price" F at $t = 1$. Similarly, the futures contract obligates the seller (an investor taking a short position) to sell a share of stock at price F at $t = 1$.

Within the simple framework that we've used many times before, with only two periods and two states at $t = 1$, the event tree below describes a stock, which sells for $q^s = 9$ at $t = 0$, $P^G = 20$ in a good state that occurs with probability $\pi = 1/2$ at $t = 1$, and $P^B = 5$ in a bad state that occurs with probability $1 - \pi = 1/2$ at $t = 1$.

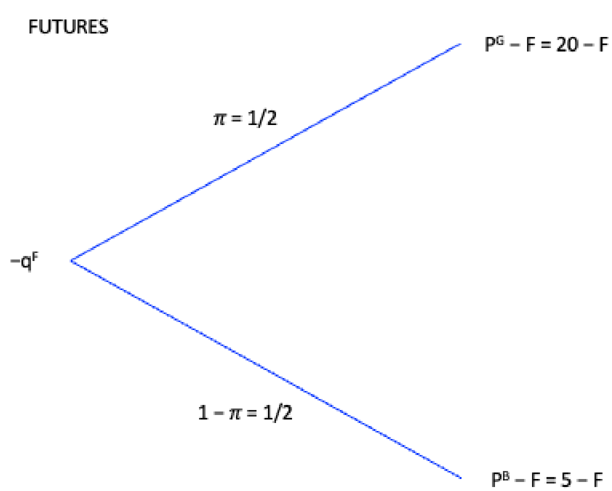


The cash flows are as seen by an investor taking a long position in the stock, so $-q^s = -9$ simply means that the investor pays $q^s = 9$ for the stock at $t = 0$, then receives either $P^G = 20$ or $P^B = 5$ depending on what happens at $t = 1$.

A bond, meanwhile, sells for $q^b = 0.9$ at $t = 0$ and pays off one no matter what at $t = 1$. The cash flows as seen by an investor taking a long position in the bond are illustrated by the event tree below.



An investor taking a long position in the futures contract pays q^F for the contract at $t = 0$. In the good state at $t = 1$, that investor is obligated to pay F for a share of stock, which he or she can sell for $P^G = 20$. Likewise, the in the bad state at $t = 1$, the investor is obligated to pay F for the share, which he or she can sell for $P^B = 5$. The event tree illustrating these cash flows is shown below. Note that if the delivery price F lies somewhere between $P^G = 20$ and $P^B = 5$, the long position in the futures contract results in a gain (a positive payment of $P^G - F = 20 - F$) in the good state and a loss (a negative payment of $P^B - F = 5 - F$) in the bad state.



In this problem, we will use a familiar no-arbitrage argument to "price" the futures contract, replicating the payoffs from the futures contract with a portfolio of the stock and the bond, then computing the cost of assembling that portfolio.

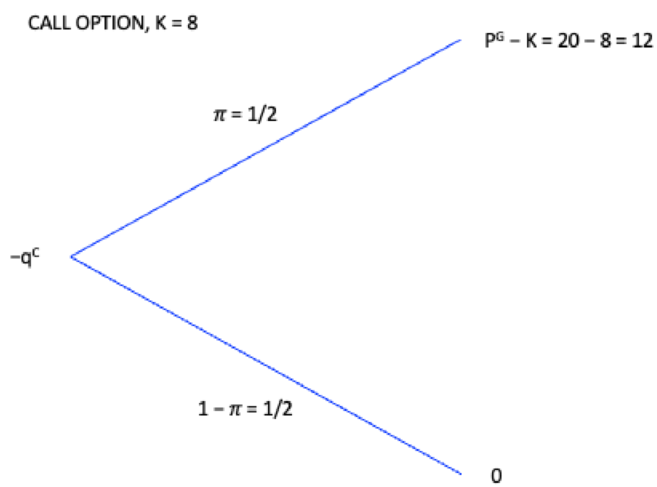
a. To begin, we need to find the portfolio consisting of s shares of stock and b bonds that will replicate the payoffs on the futures contract. Using the data on cash flows given above, write down a system of two equations in the two unknowns s and b that must hold if the payoffs from the portfolio in both the good and bad state at $t = 1$ are to match the payoffs from the futures contract at $t = 1$. Then, use this system of equations to find solutions for s and b. *Note:* Although the solution for s will turn out to be a number, the solution for b will depend on the futures delivery price F.

b. Next, find the price q^F that the futures contract must sell for at $t = 0$ if there are to be no arbitrage opportunities across the markets for the stock, bond, and futures contract. *Note:* Once again, the solution for q^F will depend on the futures delivery price F.

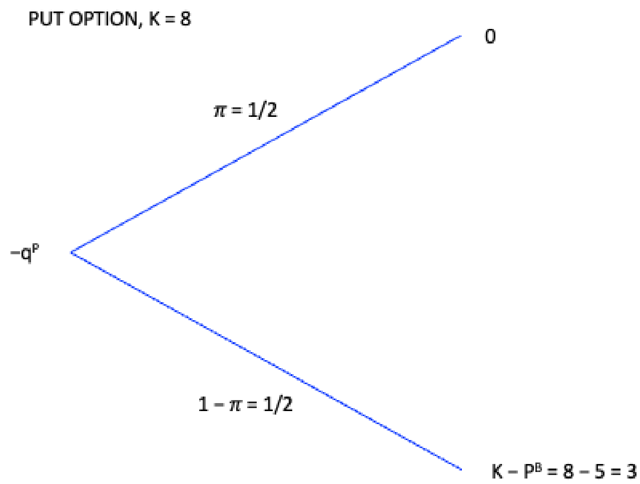
c. It turns out that, in practice, the prices quoted for futures contracts correspond not to q^F, but rather to the value of the delivery price F that makes $q^F = 0$. Although this practice might seem strange and unnecessarily complicated when you first hear about it, after some thought it makes sense. The practice means that the quoted "futures price" corresponds to the delivery price F that a buyer and seller of the futures contract agree on so that, with $q^F = 0$, no money needs to "change hands" at $t = 0$. By setting $q^F = 0$ in your solution to part (b), above, find the numerical solution for the quoted delivery price F for the futures contract.

2. Option Pricing

As discussed in class, a *call option* on the stock from question 1, above, gives the buyer (an investor taking a long position), the right but not the obligation to buy a share of stock at the "strike price" K at $t = 1$. If, in particular, $K = 8$, then the call will be "in the money" in the good state, where it is worth $P^G - K = 20 - 8 = 12$ and will be exercised, and "out of the money" in the bad state, where it is worth zero and will not be exercised. The cash flows as seen by an investor taking a long position in this call option are illustrated in the even tree below, where q^C is the price paid for the call at $t = 0$.



Another type of option on the stock, called a *put option*, gives the buyer (an investor taking a long position), the right but not the obligation to *sell* a share of stock at the strike price K at $t = 1$. If, again, $K = 8$, then the put will be out of the money in the good state, since it is better for the holder to let the put expire and sell the share at the higher market price $P^G = 20$ instead. The put will be in the money in the bad state, however, since the investor can buy the share at the market price $P^B = 5$, then exercise the option in order to sell it for the higher strike price $K = 8$, thereby earning a profit of $K - P^B = 8 - 5 = 3$. The cash flows as seen by an investor taking a long position in this put option are therefore illustrated by the event three below, where q^P is the price paid for the put at $t = 0$.



a. Find the portfolio consisting of s shares of stock and b that will replicate the payoffs on the call option with $K = 8$. Here, s and b will both be numbers. Then, find the numerical value of the price q^C that the call must sell for if there are to be no arbitrage opportunities across the markets for the stock, bond, and call option.

b. Likewise, find the portfolio consisting of s shares of stock and b that will replicate the payoffs on the put option with $K = 8$. Once again, s and b will both be numbers. Then, find the numerical value of the price q^P that the put must sell for if there are to be no arbitrage opportunities across the markets for the stock, bond, and put option.

3. Put-Call Parity

A famous result from option pricing theory is called "put-call parity," even though it involves the prices of bonds and futures contracts as well as puts and calls. Using the same notation from questions 1 and 2, above, the put-call parity condition states that

$$q^C - q^P = q^b(F - K),$$

that is, the difference between the price of a call and the price of a put must equal the price of a bond times the difference between the quoted delivery price on a futures contract and the common strike price K for the two options. Although this sounds complicated, the relationship follows from a relatively straightforward no-arbitrage argument. To see, consider two investment strategies.

Strategy one involves buying (taking a long position in) one call and "writing" (taking a short position in) one put. As shown by the event tree for the call from question 2, the long position in the call will be worth $P^G - K$ in the good state at $t = 1$ and zero in the bad state at at $t = 1$. And as implied by the event tree for the put from question 2, the short position in the put will be worth zero in the good state at $t = 1$ and $-(K - P^B) = P^B - K$ in the bad state at at $t = 1$. Therefore, strategy one pays off $P^G - K$ (a positive amount) in the good state at $t = 1$ and $P^B - K$ (a negative amount) in the bad state at $t = 1$

Strategy two involves buying (taking a long position in) one futures contract and also buying (taking a long position in) $F - K$ bonds. As shown by the event tree for the futures contract from question 1, the futures contract will be worth $P^G - F$ (a positive amount) in the good state at $t = 1$ and $P^B - F$ (a negative amount) in the bad state at $t = 1$. And as implied by the event tree for the bond from question 1, the bonds will be worth $F - K$ in both states of the world at $t = 1$. Therefore, strategy two pays off $(P^G - F) + (F - K) = P^G - K$ in the good state at $t = 1$ and $(P^B - F) + (F - K) = P^B - K$ in the bad state at $t = 1$.

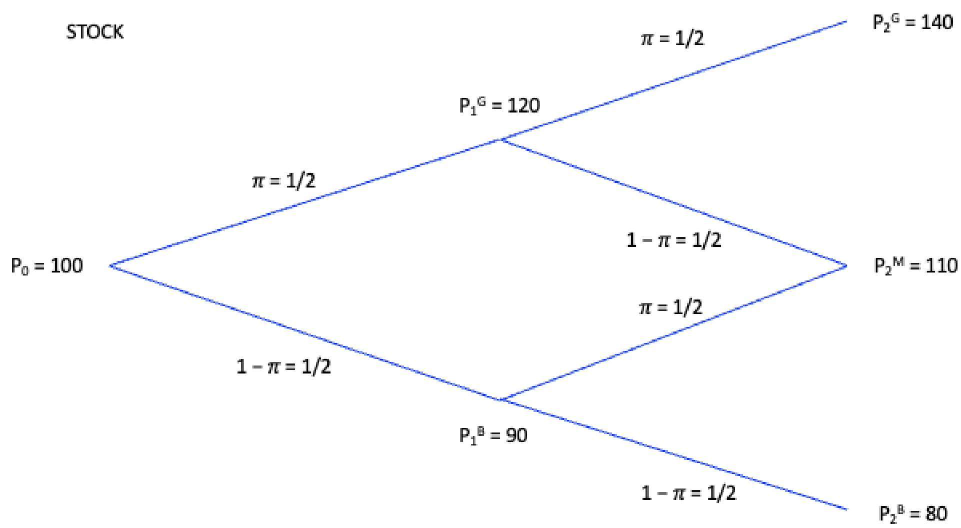
Since the two strategies yield exactly the same payoffs in each state of the work at $t = 0$, the cost of assembling the two portfolios at $t = 0$ must also be exactly the same. Otherwise, it would be possible for an investor to take a long position in the undervalued portfolio, a short position in the overvalued portfolio, and earn instantaneous, risk-free profits from the arbitrage opportunity. The cost of the portfolio required by strategy one, where the investor takes a long position in the call and a short position in the put, is $q^C - q^P$. The cost of the portfolio required by strategy two, where the investor takes a long position in the futures contract (which, as you'll recall, sets the delivery price F so as to require "no money down" or $q^F = 0$ at $t = 0$) and a long position in $F - K$ bonds, is $q^b(F - K)$. Thus, the put-call parity condition simply states that the costs of these two portfolios must be equal.

For this question, all you need to do is to use the put-call parity condition to check your answers from questions 1 and 2. Recall from question 1 that $q^b = 0.9$ and from question 2 that $K = 8$. Then take your numerical solution for F from question 1 to calculate the numerical value of $q^b(F - K)$. This number should be exactly the same as the value of $q^C - q^P$ implied by the solutions for q^C and q^P that you found in question 2.

4. Dynamic Hedging

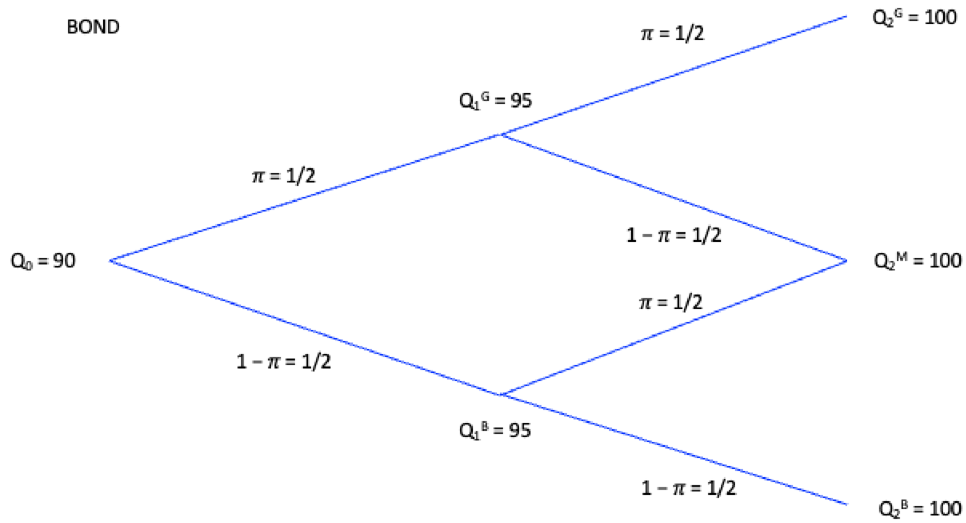
By solving this problem, you will see how, in a richer and more realistic environment where there are more than two future states of the world, traders can use a "dynamic hedging" strategy to replicate payoffs on a stock option. The strategy is "dynamic" because the trader must adjust the numbers of shares of stock and government bonds used to replicate the option's payoffs as the stock price rises or falls over time. By computing the cost of the shifting portfolio at different dates, you will also see how the price of the stock option is determined and how it, too, changes over time.

Suppose, in particular, that there are three periods: $t = 0$, $t = 1$, and $t = 2$. Suppose the price of a share of stock is initially $P_0 = 100$ in period $t = 0$ and that, in between each of the two periods that follow, the stock price either rises by 20 with probability $\pi = 1/2$ or falls by 10 with probability $1 - \pi = 1/2$. These assumptions imply that the stock price follows the pattern illustrated by the "binomial tree" shown below:



In particular, from the initial price $P_0 = 100$ in period $t = 0$, the stock price rises to $P_1^G = 120$ in a good state in period $t = 1$ but falls to $P_1^B = 90$ in a bad state in period $t = 1$. Then, if the good state occurs at $t = 1$, the stock price will rise to $P_2^G = 140$ in a good state in period $t = 2$ but fall to $P_2^M = 110$ in a medium state in period $t = 2$. And if the bad state occurs at $t = 1$, the stock price will rise to $P_2^M = 110$ in a medium state in period $t = 2$ but fall to $P_2^B = 80$ in a bad state at $t = 2$. Note that since there are two paths along the binominal tree that lead to the medium state at $t = 2$, but only one path that leads to the good state at $t = 2$ and one path that leads to the bad state $t = 2$, the medium state is more likely to occur. In particular, from the perspective of $t = 0$, the medium state at $t = 2$ will occur with probability $1/2$, while the good and bad states at $t = 2$ will each occur with probability $1/4$.

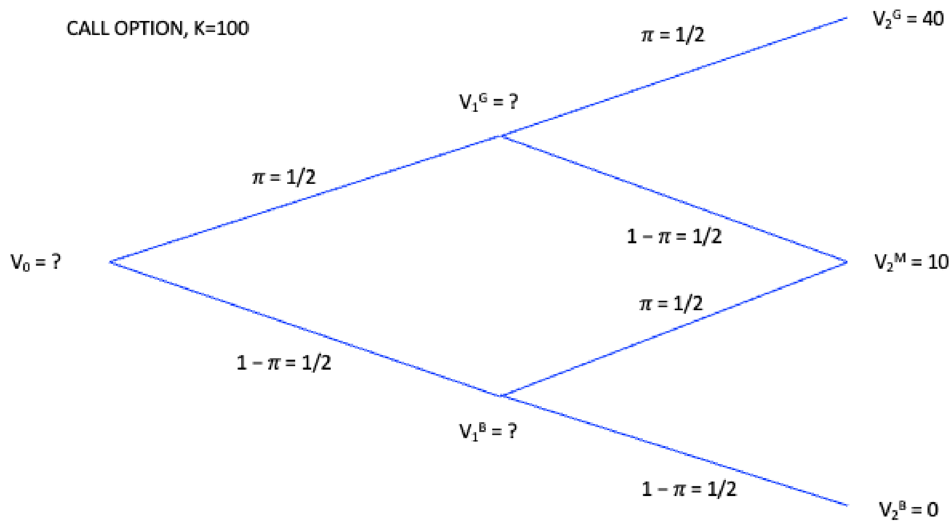
Suppose that, in the meantime, the price of a government bond rises gradually over time, so that as shown in the binominal tree below, $Q_0 = 90$ is the price of the bond at $t = 0$, $Q_1^G = 95$ and $Q_2^B = 95$ are the prices of the bond in the good and bad states at $t = 1$, and $Q_2^G = 100$, $Q_2^M = 100$, and $Q_2^B = 100$ are the prices of the bond in the good, medium, and bad states at $t = 2$.



Thus, the bond in this example is risk-free: an investor who buys it for $Q_0 = 90$ at $t = 0$ can hold it for one period and receive $Q_1^G = Q_2^B = 95$ for sure at $t = 1$ or hold it for two periods at receive $Q_2^G = Q_2^M = Q_2^B = 100$ for sure at $t = 2$. Likewise, an investor who buys the bond for 95 in either state at $t = 1$ can hold it for one period and receive 100 for sure at $t = 2$.

The stock, meanwhile, is risky: it offers a better percentage return moving from $t = 0$ to the good state at $t = 1$, moving from the good state at $t = 1$ to the good state at $t = 2$, and moving from the bad state at $t = 1$ to the medium state at $t = 2$, but exposes the investor to a loss moving from $t = 0$ to the bad state at $t = 1$, moving from the good state at $t = 1$ to the medium state at $t = 2$, and moving from the bad state at $t = 1$ to the bad state at $t = 2$.

Our goal will be to use this information about the prices of the stock and bond to "price" a call option that gives the holder the right, but not the obligation, to buy a share of stock at the strike price $K = 100$ at $t = 2$. With reference to the binomial tree for the stock, we can infer that the holder of this option will find it optimal to exercise when it is "in the money" in the good and medium states at $t = 2$ but to allow the option to expire when it is "out of the money" in the bad state at $t = 2$. We can begin constructing the binomial tree for the option itself, therefore, by noting that the option's value will be $V_2^G = 40$ in the good state at $t = 2$, $V_2^M = 10$ in the medium state at $t = 2$, and $V_2^B = 0$ in the bad state at $t = 2$:



As indicated in this same binominal tree, our task that remains is to use no arbitrage arguments to determine the price (or "value") of the option V_1^G and V_1^B in the good and bad states at $t = 1$ and the price of the option V_0 at $t = 0$.

To accomplish these goals, we will work through a process of "backwards recursion," so called because we will start by finding the value of the option in each of the two states at $t = 1$ and then use those results to determine the value of the option at $t = 0$.

a. Start by considering the situation that prevails in the good state at $t = 1$. At that time and in that state, the stock sells for $P_1^G = 120$ and the bond sells for $Q_1^G = 95$. Looking ahead to $t = 2$, the stock price can rise to $P_2^G = 140$ in the good state at $t = 2$ but can fall to $P_2^M = 110$ in the medium state at $t = 2$. In the meantime, the bond's value rises to $Q_2^G = Q_2^M = 100$ no matter what happens between $t = 1$ and $t = 2$. We want to find a portfolio consisting of s shares of stock and b bonds that will replicate the option's payoffs, equal to $V_2^G = 40$ in the good state at $t = 2$ and $V_2^M = 10$ in the medium state at $t = 2$. If we look at the problem in this way, we can see that mathematically, it takes the same form as those we've solved before. To match the option's payoff in the good state, s and b must satisfy

$$140s + 100b = 40$$

and to match the option's payoff in the medium state, s and b must satisfy

$$110s + 100b = 10.$$

Use this two-equation system to find the numerical values of s and b, the numbers of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from the good state at $t = 1$. Then, use the fact that the stock sells for $P_1^G = 120$ and the bond sells for $Q_1^G = 95$ in the good state at $t = 1$ to compute the price V_1^G of the option in the good state at $t = 1$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

b. Now consider instead the situation that prevails in the bad state at $t = 1$. At that time and in that state, the stock sells for $P_1^B = 90$ and the bond sells for $Q_1^B = 95$. Looking ahead to $t = 2$, the stock price can rise back to $P_2^M = 110$ in the medium state at $t = 2$ but can fall still further to $P_2^B = 80$ in the bad state at $t = 2$. In the meantime, the bond's value rises to $Q_2^M = Q_2^B = 100$ no matter what happens between $t = 1$ and $t = 2$. Once again, we want to find a portfolio consisting of s shares of stock and b bonds that will replicate the option's payoffs, equal to $V_2^M = 10$ in the good state at $t = 2$ and $V_2^B = 0$ in the bad state at $t = 2$. Using all of this information, write down the two equations that s and b must satisfy and use them to find the numerical values of s and b, the number of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from the bad state at $t = 1$. Then, use these values of s and b to compute the price V_1^B of the option in the bad state at $t = 1$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

c. Finally, let's step back to $t = 0$, when the stock sells for $P_0 = 100$ and the bond for $Q_0 = 90$. Looking ahead to $t = 1$, we know that the stock price will rise to $P_1^G = 120$ in the good state but fall to $P_1^B = 90$ in the bad state. We also know that the bond price will rise to $Q_1^G = Q_1^B = 95$ no matter what. Once more, we want to find a portfolio consisting of s shares of stock and b bonds to replicate the options payoffs, equal to V_1^G if we move to the good state at $t = 1$ and V_1^B if we move to the bad state at $t = 1$, where the numerical values for V_1^G and V_1^B are known from the solutions to parts (a) and (b), above. Write down the two equations that s and b must satisfy and use them to find the numerical values of s and b, the number of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from $t = 0$ to $t = 1$. Then, use these values of s and b to compute the price V_0 of the option at $t = 0$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

5. Using Options to Infer Contingent Claims Prices

In 1978, Douglas Breeden and Robert Litzenberger showed how options on the Standard & Poor's 500 stock index could be used to infer the prices of contingent claims in the real world. Recall from our discussion in class that to do this, they assumed that there are N states of the world, corresponding to different levels of the S&P500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

for some $\delta > 0$. That is, better states of the world correspond to higher levels of the S&P 500, with levels of the S&P 500 arranged on a grid with δ points between each entry.

Next, Breeden and Litzenberger showed that if one constructs a "butterfly" portfolio of call options by buying one call on the S&P 500 with strike price P^{i-1}, selling short (sometimes called "writing") two calls on the S&P 500 with strike price P^i, and buying one call on the S&P 500 with strike price P^{i+1}, then the resulting portfolio will pay off δ dollars in state i, when the S&P 500 is at level $P = P^i$, and zero otherwise. Thus, if q_o^i denotes the price of a call option with strike price P^i, no arbitrage implies that the price q_{cc}^i of a contingent claim that pays off one dollar in state i and zero otherwise can be computed as

$$q_{cc}^i = (1/\delta)(q_o^{i-1} + q_o^{i+1} - 2q_o^i).$$

The table below shows prices during the afternoon of Thursday, February 27, 2025 (just before spring break, when the S&P 500 itself stood at 5943) of call options on the S&P 500 expiring on Friday, May 16, 2025 (just before graduation day) for six strike prices on a grid that sets $\delta = 100$, taken from the "quotes dashboard" on the website of the Chicago Board Options Exchange:

S&P 500 Call Option Prices
May 16, 2025 Expiration

Strike Price	Option Price
$K = P^1 = 5800$	$q_o^1 = 295$
$K = P^2 = 5900$	$q_o^2 = 224$
$K = P^3 = 6000$	$q_o^3 = 160$
$K = P^4 = 6100$	$q_o^4 = 107$
$K = P^5 = 6200$	$q_o^5 = 64$
$K = P^6 = 6300$	$q_o^6 = 35$

Use these data, together with Breeden and Litzenberger's formula, to infer the prices on February 27 of contingent claims for the states in which the S&P 500 is at $P^2 = 5900$, $P^3 = 6000$, $P^4 = 6100$, and $P^5 = 6200$ on May 16.