Midterm Exam

ECON 337901 - Financial Economics | Peter Ireland
Boston College, Department of Economics | Spring 2024

Due Tuesday, March 26

This exam has four questions on ten pages; before you begin, please check to make sure that your copy has all four questions and all ten pages. Please note, as well, that question 1 has two parts, question 2 has three parts, question 3 has four parts, and question 4 has just one part. Each part of each question will be weighted equally in determining your overall exam score, so that question 1 is worth 20 points, question 2 is worth 30 points, question 3 is worth 40 points, and question 4 is worth 10 points, for a total of 100 points overall.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

This is an open-book exam, meaning that it is fine for you to consult your notes, material from the course and Canvas webpages, and other printed and electronic resources when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Taxes, Consumption, and Labor Supply

Consider a consumer who prefers to consume more but also prefers to work less. Suppose, in particular, that his or her preferences are described by the utility function

$$\ln(c) - \left(\frac{1}{2}\right) l^2,$$

where c denotes consumption and l denotes labor supplied. Let $w > 0$ denote the wage rate. Then the consumer's before-tax income is wl. Suppose, however, that the government taxes labor income at the constant rate T, where $1 > T \geq 0$. Then the consumer's after-tax income is $(1 - T)wl$ and his or her budget constraint is

$$(1 - T)wl \geq c.$$

a. To solve the consumer's problem, start by defining the Lagrangian, as usual, as

$$L(c, l, \lambda) = \ln(c) - \left(\frac{1}{2}\right) l^2 + \lambda[(1 - T)wl - c].$$

then derive the first-order conditions for the consumer's optimal choices of consumption c^* and labor supply l^* by differentiating the Lagrangian first with respect to c and then with respect to l and, in both cases, setting the result equal to zero.

b. Your two first-order conditions from part (a), together with the binding constraint

$$(1 - T)wl^* = c^*,$$

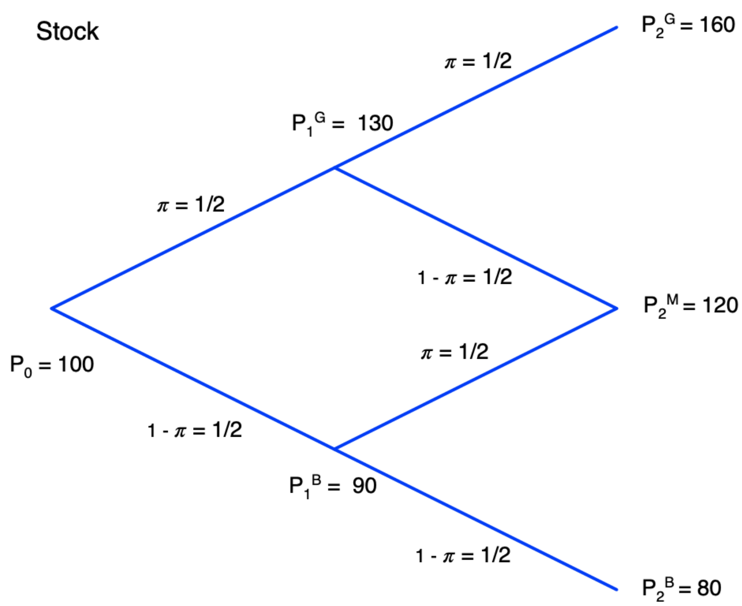
form a system of three equations in three unknowns: the optimal choices c^* and l^* and the corresponding value of λ^*. Use this system of equations to find solutions that show how c^* depends on the after-tax wage rate $(1 - T)w$ while l^* stays constant (that is, independent of both the wage and tax rates). *Note:* Although there are many ways to do this, perhaps the easiest is to use the first-order conditions to obtain expressions for c^* and l^* in terms of λ^*, substitute these expressions into the binding constraint to solve for λ^* (remembering that according to the Kuhn-Tucker theorem, the value of λ^* must be positive), then substitute this expression for λ^* back into the first-order conditions to obtain the desired solutions for c^* and l^*.

It might seem strange that, according to the solution to this problem, the consumer's optimal labor supply l^* does not depend on the wage rate or the tax rate. The reason is that, when preferences are described by the particular utility function used here, the substitution effect according to which a higher wage or lower tax rate leads the consumer to increase his or her labor supply and enjoy less leisure exactly offsets the wealth effect according to which a higher wage or lower tax rate leads the consumer to decrease his or her labor supply and enjoy more leisure.

2. Dynamic Hedging

By solving this problem, you will see how, in a richer and more realistic environment where there are more than two future states of the world, traders can use a "dynamic hedging" strategy to replicate payoffs on a stock option. The strategy is "dynamic" because the trader must adjust the numbers of shares of stock and government bonds used to replicate the option's payoffs as the stock price rises or falls over time. By computing the cost of the shifting portfolio at different dates, you will also see how the price of the stock option is determined and how it, too, changes over time.

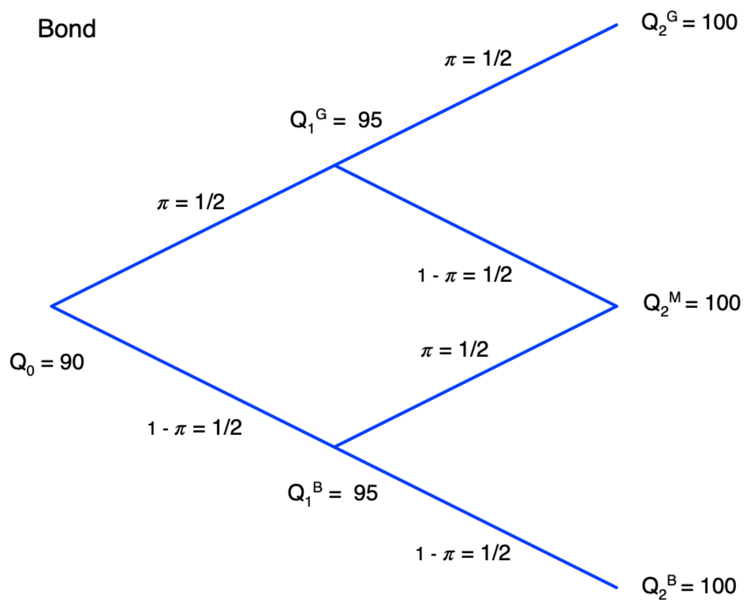
Suppose, in particular, that there are three periods: $t = 0$, $t = 1$, and $t = 2$. Suppose the price of a share of stock is initially $P_0 = 100$ in period $t = 0$ and that, in between each of the two periods that follow, the stock price either rises by 30 with probability $\pi = 1/2$ or falls by 10 with probability $1 - \pi = 1/2$. These assumptions imply that the stock price follows the pattern illustrated by the "binomial tree" shown below:



In particular, from the initial price $P_0 = 100$ in period $t = 0$, the stock price rises to $P_1^G = 130$ in a good state in period $t = 1$ but falls to $P_1^B = 90$ in a bad state in period $t = 1$. Then, if the good state occurs at $t = 1$, the stock price will rise to $P_2^G = 160$ in a good state in period $t = 2$ but fall to $P_2^M = 120$ in a medium state in period $t = 2$. And if the bad state occurs at $t = 1$, the stock price will rise to $P_2^M = 120$ in a medium state in period $t = 2$ but fall to $P_2^B = 80$ in a bad state at $t = 2$. Note that since there are two paths along the binominal tree that lead to the medium state at $t = 2$, but only one path that leads to the good state at $t = 2$ and one path that leads to the bad state $t = 2$, the medium state is more likely to occur. In particular, from the perspective of $t = 0$, the medium state at $t = 2$

will occur with probability 1/2, while the good and bad states at $t = 2$ will each occur with probability 1/4.

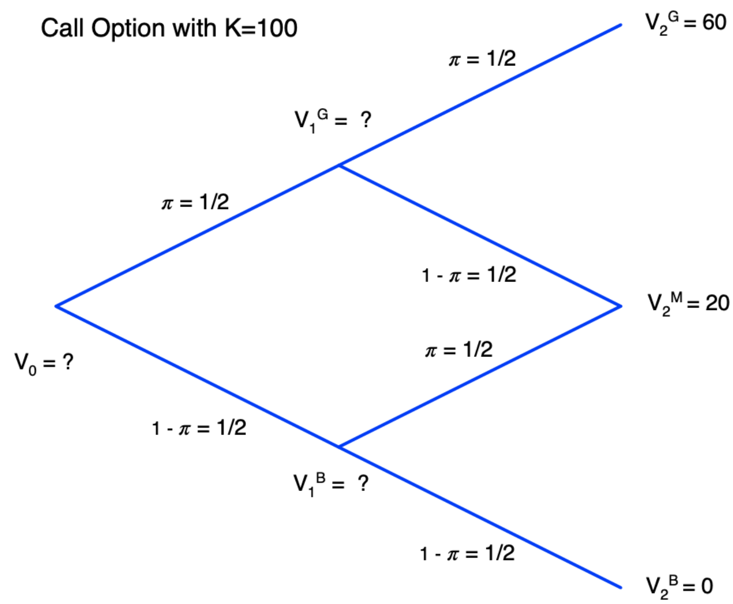
Suppose that, in the meantime, the price of a government bond rises gradually over time, so that as shown in the binominal tree below, $Q_0 = 90$ is the price of the bond at $t = 0$, $Q_1^G = 95$ and $Q_2^B = 95$ are the prices of the bond in the good and bad states at $t = 1$, and $Q_2^G = 100$, $Q_2^M = 100$, and $Q_2^B = 100$ are the prices of the bond in the good, medium, and bad states at $t = 2$.



Thus, the bond in this example is risk-free: an investor who buys it for $Q_0 = 90$ at $t = 0$ can hold it for one period and receive $Q_1^G = Q_2^B = 95$ for sure at $t = 1$ or hold it for two periods at receive $Q_2^G = Q_2^M = Q_2^B = 100$ for sure at $t = 2$. Likewise, an investor who buys the bond for 95 in either state at $t = 1$ can hold it for one period and receive 100 for sure at $t = 2$.

The stock, meanwhile, is risky: it offers a better percentage return moving from $t = 0$ to the good state at $t = 1$, moving from the good state at $t = 1$ to the good state at $t = 2$, and moving from the bad state at $t = 1$ to the medium state at $t = 2$, but exposes the investor to a loss moving from $t = 0$ to the bad state at $t = 1$, moving from the good state at $t = 1$ to the medium state at $t = 2$, and moving from the bad state at $t = 1$ to the bad state at $t = 2$.

Our goal will be to use this information about the prices of the stock and bond to "price" a call option that gives the holder the right, but not the obligation, to buy a share of stock at the strike price $K = 100$ at $t = 2$. With reference to the binomial tree for the stock, we can infer that the holder of this option will find it optimal to exercise when it is "in the money" in the good and medium states at $t = 2$ but to allow the option to expire when it is "out of the money" in the bad state at $t = 2$. We can begin constructing the binomial tree for the option itself, therefore, by noting that the option's value will be $V_2^G = 60$ in the good state at $t = 2$, $V_2^M = 20$ in the medium state at $t = 2$, and $V_2^B = 0$ in the bad state at $t = 2$:



As indicated in this same binominal tree, our task that remains is to use no arbitrage arguments to determine the price (or "value") of the option V_1^G and V_1^B in the good and bad states at $t = 1$ and the price of the option V_0 at $t = 0$.

To accomplish these goals, we will work through a process of "backwards recursion," so called because we will start by finding the value of the option in each of the two states at $t = 1$ and then use those results to determine the value of the option at $t = 0$.

a. Start by considering the situation that prevails in the good state at $t = 1$. At that time and in that state, the stock sells for $P_1^G = 130$ and the bond sells for $Q_1^G = 95$. Looking ahead to $t = 2$, the stock price can rise to $P_2^G = 160$ in the good state at $t = 2$ but can fall to $P_2^M = 120$ in the medium state at $t = 2$. In the meantime, the bond's value rises to $Q_2^G = Q_2^M = 100$ no matter what happens between $t = 1$ and $t = 2$. We want to find a portfolio consisting of s shares of stock and b bonds that will replicate the option's payoffs, equal to $V_2^G = 60$ in the good state at $t = 2$ and $V_2^M = 20$ in the medium state at $t = 2$. If we look at the problem in this way, we can see that mathematically, it takes the same form as those we've solved before. To match the option's payoff in the good state, s and b must satisfy

$$160s + 100b = 60$$

and to match the option's payoff in the medium state, s and b must satisfy

$$120s + 100b = 20.$$

Use this two-equation system to find the numerical values of s and b, the numbers of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from the good state at $t = 1$. Then, use the fact that the stock sells for $P_1^G = 130$ and the bond sells for $Q_1^G = 95$ in the good state at $t = 1$ to compute the price V_1^G of the option in the good state at $t = 1$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

b. Now consider instead the situation that prevails in the bad state at $t = 1$. At that time and in that state, the stock sells for $P_1^B = 90$ and the bond sells for $Q_1^B = 95$. Looking ahead to $t = 2$, the stock price can rise back to $P_2^M = 120$ in the medium state at $t = 2$ but can fall still further to $P_2^B = 80$ in the bad state at $t = 2$. In the meantime, the bond's value rises to $Q_2^M = Q_2^B = 100$ no matter what happens between $t = 1$ and $t = 2$. Once again, we want to find a portfolio consisting of s shares of stock and b bonds that will replicate the option's payoffs, equal to $V_2^M = 20$ in the good state at $t = 2$ and $V_2^B = 0$ in the bad state at $t = 2$. Using all of this information, write down the two equations that s and b must satisfy and use them to find the numerical values of s and b, the number of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from the bad state at $t = 1$. Then, use these values of s and b to compute the price V_1^B of the option in the bad state at $t = 1$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

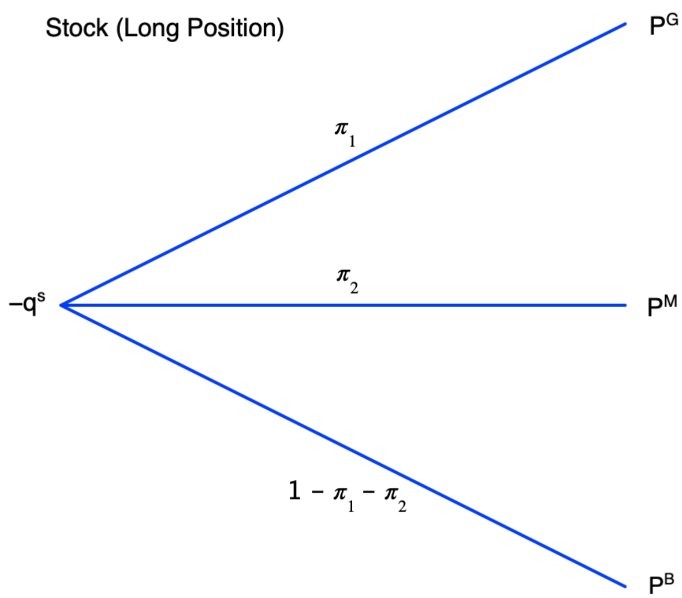
c. Finally, let's step back to $t = 0$, when the stock sells for $P_0 = 100$ and the bond for $Q_0 = 90$. Looking ahead to $t = 1$, we know that the stock price will rise to $P_1^G = 130$ in the good state but fall to $P_1^B = 90$ in the bad state. We also know that the bond price will rise to $Q_1^G = Q_1^B = 95$ no matter what. Once more, we want to find a portfolio consisting of s shares of stock and b bonds to replicate the options payoffs, equal to V_1^G if we move to the good state at $t = 1$ and V_1^B if we move to the bad state at $t = 1$, where the numerical values for V_1^G and V_1^B are known from the solutions to parts (a) and (b), above. Write down the two equations that s and b must satisfy and use them to find the numerical values of s and b, the number of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from $t = 0$ to $t = 1$. Then, use these values of s and b to compute the price V_0 of the option at $t = 0$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

3. Futures Pricing

Now let's move back towards the simpler framework that we've used many times before, with only two periods: today ($t = 0$) and next year ($t = 1$). Suppose, however, that there are three possible states next year: a good state that occurs with probability π_1, a medium state that occurs with probability π_2, and a bad state that occurs with probability $1 - \pi_1 - \pi_2$.

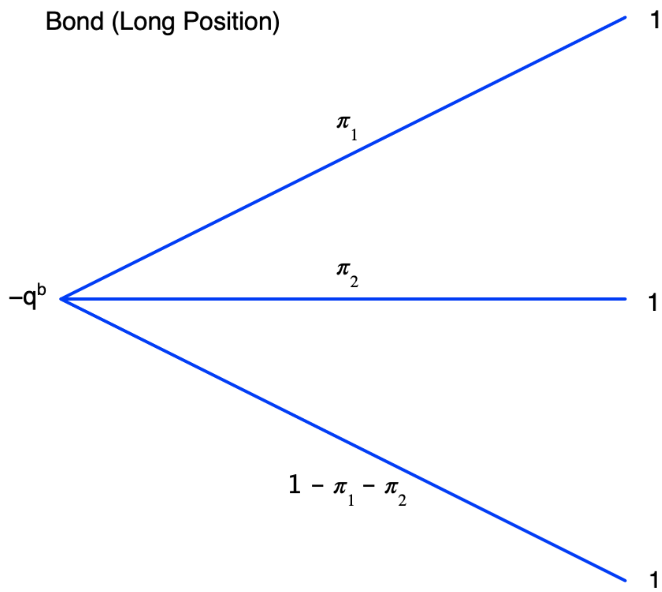
As discussed in the introductory video for this class, a *futures contract* is similar to a stock option, except that it *obligates* the buyer (an investor taking a long position) to buy a share of stock at the "delivery price" F at $t = 1$. Similarly, the futures contract obligates the seller (an investor taking a short position) to sell a share of stock at price F at $t = 1$.

The event tree below describes a stock, which sells for q^s at $t = 0$, P^G in a good state at $t = 1$, P^M in the medium state at $t = 1$, and P^B in the bad state that occurs at $t = 1$.

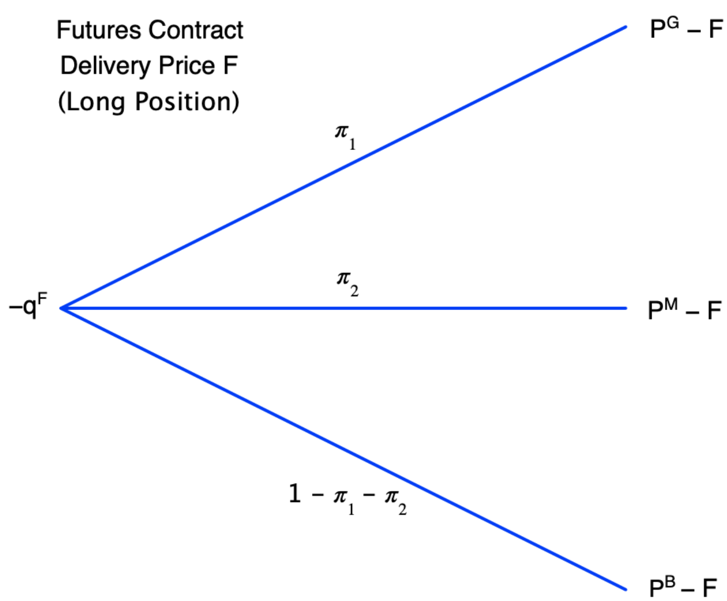


The cash flows are as seen by an investor taking a long position in the stock, so $-q^s$ simply means that the investor pays q^s for the stock at $t = 0$, then receives either P^G, P^M, or P^B depending on what happens at $t = 1$.

A bond, meanwhile, sells for q^b at $t = 0$ and pays off one no matter what at $t = 1$. The cash flows as seen by an investor taking a long position in the bond are illustrated by the event tree below.



An investor taking a long position in the futures contract pays q^F for the contract at $t = 0$. At $t = 1$, that investor is obligated to pay F for a share of stock, which he or she can sell for P^G, P^M, or P^B, depending on the state. The event tree illustrating these cash flows is shown below.



In this problem, we will use a no-arbitrage argument to "price" the futures contract and to see, in particular, how the futures price depends on the prices of the stock and the bond. It turns out that, even with more than two states next year, we won't need to work through a dynamic hedging argument to do this.

a. To begin, consider using a portfolio consisting of s shares of stock and b bonds to replicate the payoffs on the futures contract in the good and bad states, ignoring the medium state for now. The stock pays off P^G in the good state and P^B in the bad. The bond pays off 1 no matter what. The futures contract pays off $P^G - F$ in the good state and $P^B - F$ in the bad. So s and b must satisfy

$$P^G s + b = P^G - F$$

and

$$P^B s + b = P^B - F.$$

Solve this system of two equations to find the values of s and b needed to replicate the futures contract's payoffs in the good and bad state.

b. The portfolio of the stock and bond you found in answering part (a) is designed to match the future's contract's payoffs in the good and bad state. But what is the payoff from this portfolio in the medium state? How does it compare to the payoff from the futures contract in the medium state?

c. In answering the questions in part (a) and (b), you should have discovered that the same portfolio of the stock and bond replicates the payoffs from the futures contract in all three states. This special feature is what makes futures pricing somewhat easier that option pricing. Now compute the cost today (at $t = 0$) of assembling the portfolio of the stock and bond from part (a). If there no arbitrage opportunities across the markets for the stock, bond, and futures contract, this cost will also equal the price q^F of the futures contract today.

d. It happens that in practice, the prices quoted for futures contracts correspond not to q^F, but rather to the value of the delivery price F that makes $q^F = 0$. Although this practice might seem strange and unnecessarily complicated when you first hear about it, after some thought it makes sense. The practice means that the quoted "futures price" corresponds to the delivery price F that a buyer and seller of the futures contract agree on so that, with $q^F = 0$, no money needs to "change hands" at $t = 0$. By setting $q^F = 0$ using your solution to part (c), find a solution that shows how the quoted delivery price F will depend on the prices q^s and q^b of the stock and bond at $t = 0$. Remember, again from the introductory video, that over the course of a trading day, when interest rates and hence bond prices do not change much, the price of the Standard & Poor's 500 exchange traded fund (corresponding to q^s in this example) and the price of the Standard & Poor's 500 futures contract (corresponding to F) move up and down together. Your solution should show that, in general, so long as q^b remains constant, F and q^s will move up and down together.

4. Using Options to Infer Contingent Claims Prices

In 1978, Douglas Breeden and Robert Litzenberger showed how options on the Standard & Poor's 500 stock index could be used to infer the prices of contingent claims in the real world. Recall from our discussions in class that to do this, they assumed that there are N states of the world, corresponding to different levels of the S&P500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

for some $\delta > 0$. That is, better states of the world correspond to higher levels of the S&P 500, with levels of the S&P 500 arranged on a grid with δ points between each entry.

Next, Breeden and Litzenberger showed that if one constructs a "butterfly" portfolio of call options by buying one call on the S&P 500 with strike price P^{i-1}, selling short (sometimes called "writing") two calls on the S&P 500 with strike price P^i, and buying one call on the S&P 500 with strike price P^{i+1}, then the resulting portfolio will pay off δ dollars in state i, when the S&P 500 is at level $P = P^i$, and zero otherwise. Thus, if q_o^i denotes the price of a call option with strike price P^i, no arbitrage implies that the price q_{cc}^i of a contingent claim that pays off one dollar in state i and zero otherwise can be computed as

$$q_{cc}^i = (1/\delta)(q_o^{i-1} + q_o^{i+1} - 2q_o^i).$$

The table below shows prices during the morning of Wednesday, February 28, 2024 (just before spring break, when the S&P 500 itself stood at 5064) of call options on the S&P 500 expiring on Friday, May 17, 2024 (just before graduation day) for six strike prices on a grid that sets $\delta = 100$, taken from the "quotes dashboard" on the website of the Chicago Board Options Exchange:

S&P 500 Call Option Prices
May 17, 2024 Expiration

Strike Price	Option Price
$K = P^1 = 4800$	$q_o^1 = 344$
$K = P^2 = 4900$	$q_o^2 = 258$
$K = P^3 = 5000$	$q_o^3 = 182$
$K = P^4 = 5100$	$q_o^4 = 118$
$K = P^5 = 5200$	$q_o^5 = 70$
$K = P^6 = 5300$	$q_o^6 = 37$

Use these data, together with Breeden and Litzenberger's formula, to infer the prices on February 28 of contingent claims for the states in which the S&P 500 is at $P^2 = 4900$, $P^3 = 5000$, $P^4 = 5100$, and $P^5 = 5200$ on May 17.