Midterm Exam

ECON 337901 - Financial Economics
Boston College, Department of Economics

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Due Tuesday, March 28

This exam has four questions on eight pages; before you begin, please check to make sure that your copy has all four questions and all eight pages. Please note, as well, that question 1 has two parts, question 2 has four parts, question 3 has three parts, and question 4 has just one part. Each part of each question will be weighted equally in determining your overall exam score, so that question 1 is worth 20 points, question 2 is worth 40 points, question 3 is worth 30 points, and question 4 is worth 10 points, for a total of 100 points overall.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

This is an open-book exam, meaning that it is fine for you to consult your notes, material from the course and Canvas webpages, and other printed and electronic resources when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Utility Maximization

Consider a consumer who uses his or her income Y to purchase c_a apples at the price of p_a per apple and c_b bananas at the price of p_b per banana, subject to the budget constraint

$$Y \geq p_a c_a + p_b c_b,$$

where income Y and prices p_a and p_b are all positive numbers and where the additional restriction that $p_b < Y$ will guarantee, when you solve the problem below, that the consumer buys at least some bananas as well as apples.

Suppose that the consumer's preferences over apples and bananas are described by the utility function

$$\ln(c_a) + c_b,$$

where ln is the natural logarithm. Then the consumer solves the problem

$$\max_{c_a, c_b} \ \ln(c_a) + c_b \text{ subject to } Y \geq p_a c_a + p_b c_b.$$

a. To solve the consumer's problem, start by defining the Lagrangian, as usual, as

$$L(c_a, c_b, \lambda) = \ln(c_a) + c_b + \lambda(Y - p_a c_a - p_b c_b).$$

Now derive the two first-order conditions for the consumer's optimal choices c_a^* and c_b^* by differentiating the Lagrangian first with respect to c_a holding c_b constant and then with respect to c_b holding c_a constant and, in both cases, setting the result equal to zero. In deriving these first-order conditions, note that the derivative of the utility function with respect to c_a equals $1/c_a$ and that the derivative of the utility function with respect to c_b equals 1.

b. Your two first-order conditions from part (a), together with the binding constraint

$$Y = p_a c_a^* + p_b c_b^*,$$

form a system of three equations in three unknowns: the optimal choices c_a^* and c_b^* and the corresponding value of λ^*. Use this system to find solutions that show how, when the consumer's preferences are described by the utility function used here, the optimal c_a^* depends on p_a and p_b and the optimal c_b^* depends on Y and p_b. *Note:* Although there are many ways to do this, probably the easiest is to use the first-order condition for c_b^* to find the solution for λ^*, substitute this solution for λ^* into the first-order condition for c_a^* to find the solution for c_a^*, and then substitute this solution for c_a^* into the binding constraint to find the solution for c_b^*.

2. Volatility: Its Effects on Stock and Option Prices

It seems natural to assume that when a risky stock's future price becomes more volatile, that stock becomes less attractive to investors so that, in equilibrium, its price falls. But what happens to the price of a call option when the underlying stock price becomes more volatile? The examples in this problem will help us see.

Consider the simple framework that we've used many times before, with two periods, $t = 0$ and $t = 1$, and two possible states at $t = 1$: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$.

Across all four examples below, assume that a contingent claim for the good state sells for price $q^G = 0.3$ at $t = 0$ and pays off one in the good state and zero in the bad state at $t = 1$. A contingent claim for the bad state sells for price $q^B = 0.6$ at $t = 0$ and pays off one in the bad state and zero in the good state at $t = 1$.

a. To start, consider a stock that will have price $P^G = 10$ in the good state and price $P^B = 5$ in the bad state at $t = 1$. If there are no arbitrage opportunities across the markets for contingent claims and stocks, what will the price q^s of this stock be at $t = 0$?

b. Next, consider a call option that gives the holder the right, but not the obligation, to buy one share of the stock described in part (a) at the strike price $K = 7$ at $t = 1$. If there are no arbitrage opportunities across the markets for contingent claims, stocks, and options, what will the price q^o of this option be at $t = 0$?

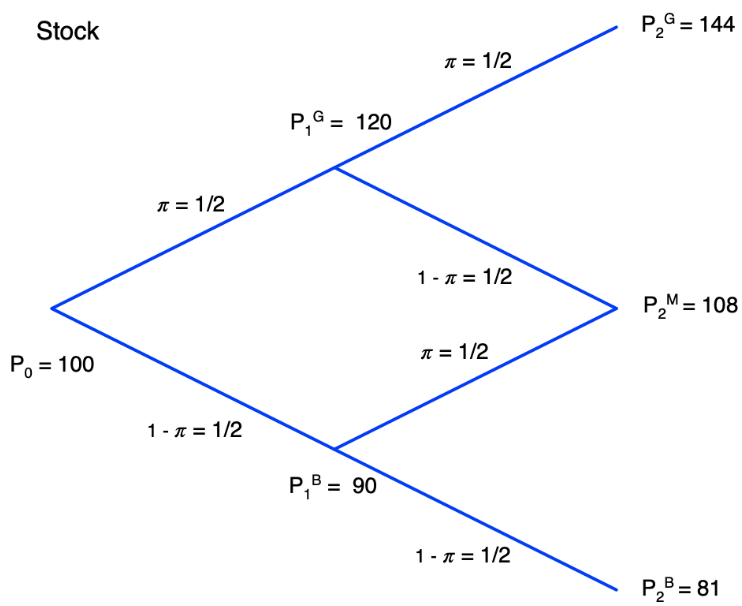
c. Now suppose instead that the stock from part (a) will have price $P^G = 12$ in the good state and price $P^B = 3$ in the bad state at $t = 1$. The stock is now riskier, in the sense that the volatility (or in the terms of formal statistics, the variance or standard deviation) of its price at $t = 1$ has increased. If there are no arbitrage opportunities across the markets for contingent claims and stocks, what will the price q^s of this stock be at $t = 0$? Compared to what you found in part (a), is the stock price higher or lower now that volatility has increased?

d. Finally, consider the call option that gives the holder the right, but not the obligation, to buy one share of the stock described in part (c) at the strike price $K = 7$ at $t = 1$. If there are no arbitrage opportunities across the markets for contingent claims, stocks, and options, what will the price q^o of this option be at $t = 0$? Compared to what you found in part (b), is the option price higher or lower now that the stock's price volatility has increased?

3. Dynamic Hedging

By solving this problem, you will see how, in a richer and more realistic environment where there are more than two future states of the world, traders can use a "dynamic hedging" strategy to replicate payoffs on a stock option. The strategy is "dynamic" because the trader must adjust the numbers of shares of stock and government bonds used to replicate the option's payoffs as the stock price rises or falls over time. By computing the cost of the shifting portfolio at different dates, you will also see how the price of the stock option is determined and how it, too, changes over time.

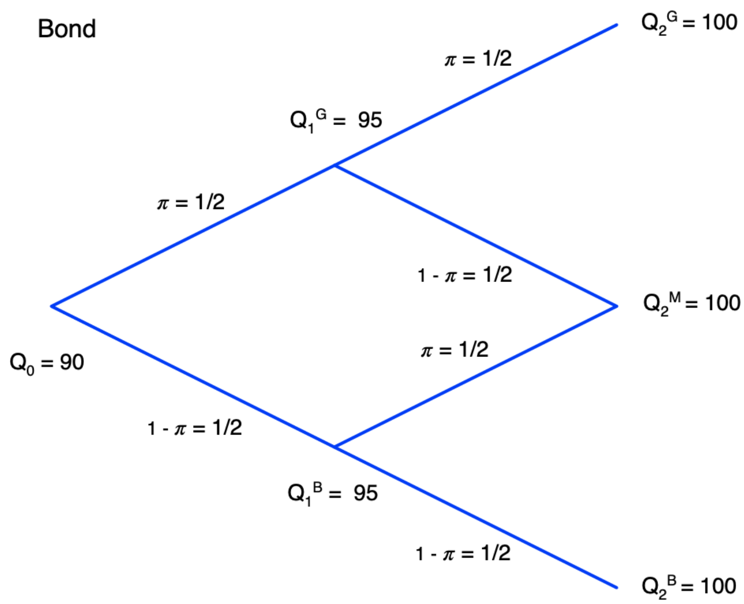
Suppose, in particular, that there are three periods: $t = 0$, $t = 1$, and $t = 2$. Suppose the price of a share of stock is initially $P_0 = 100$ in period $t = 0$ and that, in between each of the two periods that follow, the stock price either rises by 20 percent with probability $\pi = 1/2$ or falls by 10 percent with probability $1 - \pi = 1/2$. These assumptions imply that the stock price follows the pattern illustrated by the "binomial tree" shown below:



In particular, from the initial price $P_0 = 100$ in period $t = 0$, the stock price rises to $P_1^G = 120$ in a good state in period $t = 1$ but falls to $P_1^B = 90$ in a bad state in period $t = 1$. Then, if the good state occurs at $t = 1$, the stock price will rise to $P_2^G = 144$ in a good state in period $t = 2$ but fall to $P_2^M = 108$ in a medium state in period $t = 2$. And if the bad state occurs at $t = 1$, the stock price will rise to $P_2^M = 108$ in a medium state in period $t = 2$ but fall to $P_2^B = 81$ in a bad state at $t = 2$. Note that since there are two paths along the binominal tree that lead to the medium state at $t = 2$, but only one path that leads to the good state at $t = 2$ and one path that leads to the bad state $t = 2$, the medium state is more likely to occur. In particular, from the perspective of $t = 0$, the medium state at $t = 2$

will occur with probability 1/2, while the good and bad states at $t = 2$ will each occur with probability 1/4.

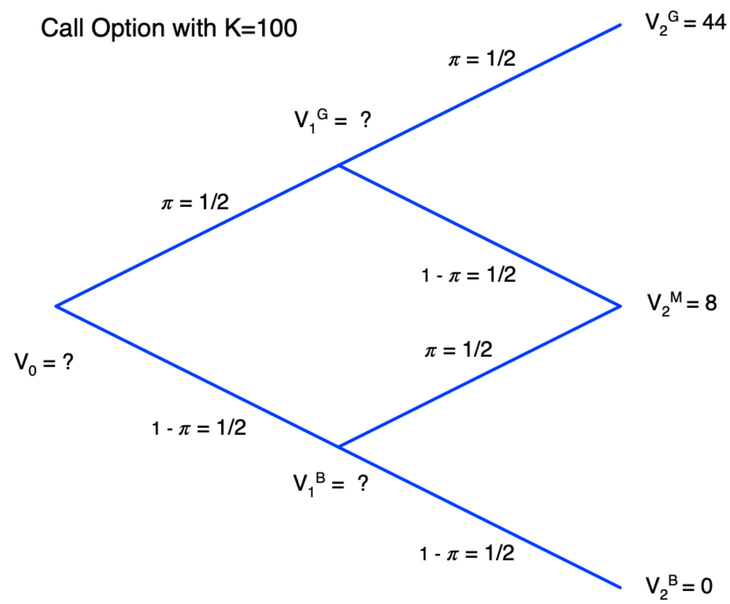
Suppose that, in the meantime, the price of a government bond rises gradually over time, so that as shown in the binominal tree below, $Q_0 = 90$ is the price of the bond at $t = 0$, $Q_1^G = 95$ and $Q_2^B = 95$ are the prices of the bond in the good and bad states at $t = 1$, and $Q_2^G = 100$, $Q_2^M = 100$, and $Q_2^B = 100$ are the prices of the bond in the good, medium, and bad states at $t = 2$.



Thus, the bond in this example is risk-free: an investor who buys it for $Q_0 = 90$ at $t = 0$ can hold it for one period and receive $Q_1^G = Q_2^B = 95$ for sure at $t = 1$ or hold it for two periods at receive $Q_2^G = Q_2^M = Q_2^B = 100$ for sure at $t = 2$. Likewise, an investor who buys the bond for 95 in either state at $t = 1$ can hold it for one period and receive 100 for sure at $t = 2$.

The stock, meanwhile, is risky: it offers a better percentage return moving from $t = 0$ to the good state at $t = 1$, moving from the good state at $t = 1$ to the good state at $t = 2$, and moving from the bad state at $t = 1$ to the medium state at $t = 2$, but exposes the investor to a loss moving from $t = 0$ to the bad state at $t = 1$, moving from the good state at $t = 1$ to the medium state at $t = 2$, and moving from the bad state at $t = 1$ to the bad state at $t = 2$.

Our goal will be to use this information about the prices of the stock and bond to "price" a call option that gives the holder the right, but not the obligation, to buy a share of stock at the strike price $K = 100$ at $t = 2$. With reference to the binomial tree for the stock, we can infer that the holder of this option will find it optimal to exercise when it is "in the money" in the good and medium states at $t = 2$ but to allow the option to expire when it is "out of the money" in the bad state at $t = 2$. We can begin constructing the binomial tree for the option itself, therefore, by noting that the option's value will be $V_2^G = 44$ in the good state at $t = 2$, $V_2^M = 8$ in the medium state at $t = 2$, and $V_2^B = 0$ in the bad state at $t = 2$:



As indicated in this same binominal tree, our task that remains is to use no arbitrage arguments to determine the price (or "value") of the option V_1^G and V_1^B in the good and bad states at $t = 1$ and the price of the option V_0 at $t = 0$.

To accomplish these goals, we will work through a process of "backwards recursion," so called because we will start by finding the value of the option in each of the two states at $t = 1$ and then use those results to determine the value of the option at $t = 0$.

a. Start by considering the situation that prevails in the good state at $t = 1$. At that time and in that state, the stock sells for $P_1^G = 120$ and the bond sells for $Q_1^G = 95$. Looking ahead to $t = 2$, the stock price can rise to $P_2^G = 144$ in the good state at $t = 2$ but can fall to $P_2^M = 108$ in the medium state at $t = 2$. In the meantime, the bond's value rises to $Q_2^G = Q_2^M = 100$ no matter what happens between $t = 1$ and $t = 2$. We want to find a portfolio consisting of s shares of stock and b bonds that will replicate the option's payoffs, equal to $V_2^G = 44$ in the good state at $t = 2$ and $V_2^M = 8$ in the medium state at $t = 2$. If we look at the problem in this way, we can see that mathematically, it takes the same form as those we've solved before. To match the option's payoff in the good state, s and b must satisfy

$$144s + 100b = 44$$

and to match the option's payoff in the medium state, s and b must satisfy

$$108s + 100b = 8.$$

Use this two-equation system to find the numerical values of s and b, the numbers of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from the good state at $t = 1$. Then, use the fact that the stock sells for $P_1^G = 120$ and the bond sells for $Q_1^G = 95$ in the good state at $t = 1$ to compute the price V_1^G of the option in the good state at $t = 1$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

b. Now consider instead the situation that prevails in the bad state at $t = 1$. At that time and in that state, the stock sells for $P_1^B = 90$ and the bond sells for $Q_1^B = 95$. Looking ahead to $t = 2$, the stock price can rise back to $P_2^M = 108$ in the medium state at $t = 2$ but can fall still further to $P_2^B = 81$ in the bad state at $t = 2$. In the meantime, the bond's value rises to $Q_2^M = Q_2^B = 100$ no matter what happens between $t = 1$ and $t = 2$. Once again, we want to find a portfolio consisting of s shares of stock and b bonds that will replicate the option's payoffs, equal to $V_2^M = 8$ in the good state at $t = 2$ and $V_2^B = 0$ in the bad state at $t = 2$. Using all of this information, write down the two equations that s and b must satisfy and use them to find the numerical values of s and b, the number of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from the bad state at $t = 1$. Then, use these values of s and b to compute the price V_1^B of the option in the bad state at $t = 1$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

c. Now, let's step back to $t = 0$, when the stock sells for $P_0 = 100$ and the bond for $Q_0 = 90$. Looking ahead to $t = 1$, we know that the stock price will rise to $P_1^G = 120$ in the good state but fall to $P_1^B = 90$ in the bad state. We also know that the bond price will rise to $Q_1^G = Q_1^B = 95$ no matter what. Once more, we want to find a portfolio consisting of s shares of stock and b bonds to replicate the options payoffs, equal to V_1^G if we move to the good state at $t = 1$ and V_1^B if we move to the bad state at $t = 1$, where the numerical values for V_1^G and V_1^B are known from the solutions to parts (a) and (b), above. Write down the two equations that s and b must satisfy and use them to find the numerical values of s and b, the number of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from $t = 0$ to $t = 1$. Then, use these values of s and b to compute the price V_0 of the option at $t = 0$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

4. Using Options to Infer Contingent Claims Prices

In 1978, Douglas Breeden and Robert Litzenberger showed how options on the Standard & Poor's 500 stock index could be used to infer the prices of contingent claims in the real world. Recall from our discussions in class that to do this, they assumed that there are N states of the world, corresponding to different levels of the S&P500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

for some $\delta > 0$. That is, better states of the world correspond to higher levels of the S&P 500, with levels of the S&P 500 arranged on a grid with δ points between each entry.

Next, Breeden and Litzenberger showed that if one constructs a "butterfly" portfolio of call options by buying one call on the S&P 500 with strike price P^{i-1}, selling short (sometimes called "writing") two calls on the S&P 500 with strike price P^i, and buying one call on the S&P 500 with strike price P^{i+1}, then the resulting portfolio will pay off δ dollars in state i, when the S&P 500 is at level $P = P^i$, and zero otherwise. Thus, if q_o^i denotes the price of a call option with strike price P^i, no arbitrage implies that the price q_{cc}^i of a contingent claim that pays off one dollar in state i and zero otherwise can be computed as

$$q_{cc}^i = (1/\delta)(q_o^{i-1} + q_o^{i+1} - 2q_o^i).$$

The table below shows prices during the afternoon of Wednesday, March 1, 2023 (just before spring break, when the S&P 500 itself stood at 3955) of call options on the S&P 500 expiring on Friday, May 19, 2023 (just before graduation day) for six strike prices on a grid that sets $\delta = 100$, taken from the "quotes dashboard" on the website of the Chicago Board Options Exchange:

S&P 500 Call Option Prices
May 19, 2023 Expiration

Strike Price	Option Price
$K = P^1 = 3700$	$q_o^1 = 335$
$K = P^2 = 3800$	$q_o^2 = 259$
$K = P^3 = 3900$	$q_o^3 = 190$
$K = P^4 = 4000$	$q_o^4 = 129$
$K = P^5 = 4100$	$q_o^5 = 79$
$K = P^6 = 4200$	$q_o^6 = 43$

Use these data, together with Breeden and Litzenberger's formula, to infer the prices on March 8 of contingent claims for the states in which the S&P 500 is at $P^2 = 3800$, $P^3 = 3900$, $P^4 = 4000$, and $P^5 = 4100$ on May 19.