Midterm Exam

ECON 337901 - Financial Economics
Boston College, Department of Economics

Peter Ireland
Spring 2021

Due Thursday, April 1

This exam has two questions on seven pages; before you begin, please check to make sure that your copy has both questions and all seven pages. Please note, as well, that question one has three parts while question 2 has just one part. Each part of each question will be weighted equally in determining your overall exam score, so that question 1 is worth 75 points in total, question 2 is worth 25 points in total, and the sum overall is 100 points.

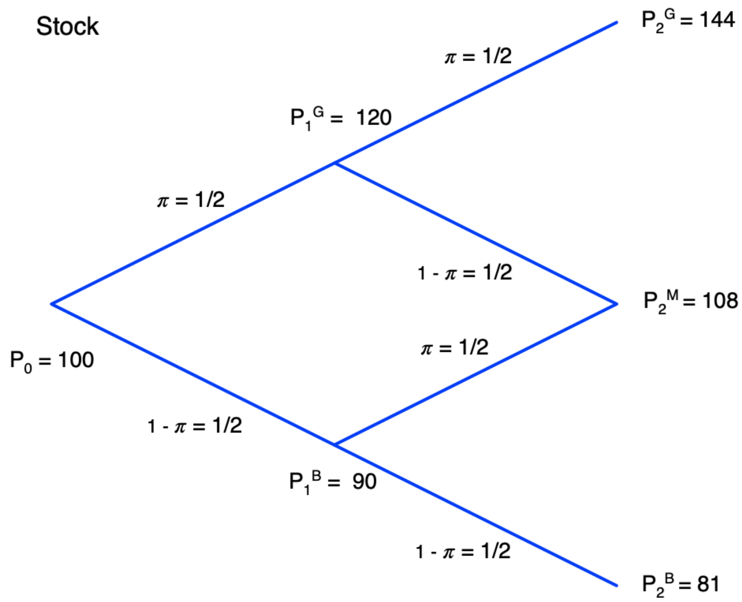
Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

This is an open-book exam, meaning that it is fine for you to consult your notes and all material from the course and Canvas webpages when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

1. Dynamic Hedging

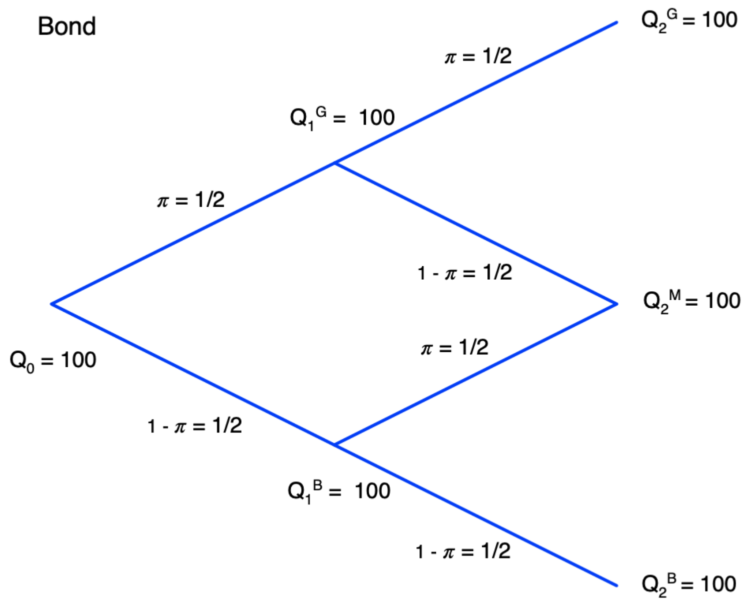
By solving this problem, you will see how, in a richer and more realistic environment where there are more than two future states of the world, traders can use a "dynamic hedging" strategy to replicate payoffs on a stock option. The strategy is "dynamic" because the trader must adjust the numbers of shares of stock and government bonds used to replicate the option's payoffs as the stock price rises or falls over time. By computing the cost of the shifting portfolio at different dates, you will also see how the price of the stock option is determined and how it, too, changes over time.

Suppose, in particular, that there are three periods: $t = 0$, $t = 1$, and $t = 2$. Suppose the price of a share of stock is initially $P_0 = 100$ in period $t = 0$ and that, in between each of the two periods that follow, the stock price either rises by 20 percent with probability $\pi = 1/2$ or falls by 10 percent with probability $1 - \pi = 1/2$. These assumptions imply that the stock price follows the pattern illustrated by the "binomial tree" shown below:



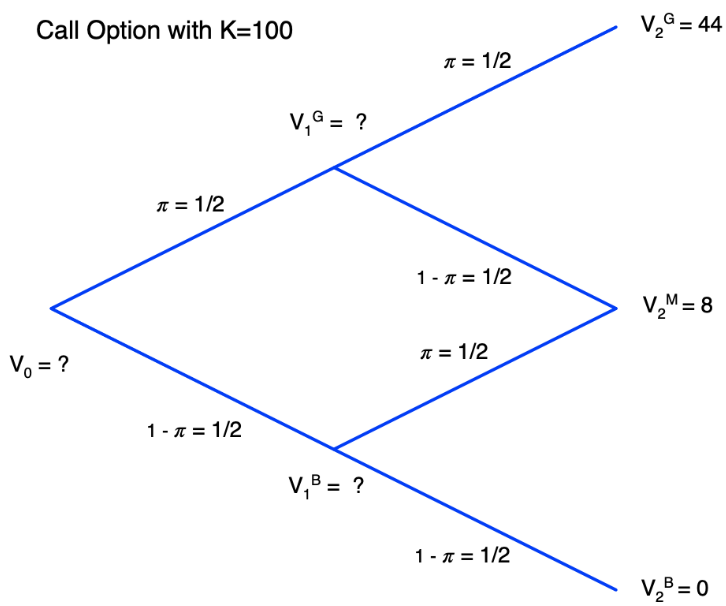
In particular, from the initial price $P_0 = 100$ in period $t = 0$, the stock price rises to $P_1^G = 120$ in a good state in period $t = 1$ but falls to $P_1^B = 90$ in a bad state in period $t = 1$. Then, if the good state occurs at $t = 1$, the stock price will rise to $P_2^G = 144$ in a good state in period $t = 2$ but fall to $P_2^M = 108$ in a medium state in period $t = 2$. And if the bad state occurs at $t = 1$, the stock price will rise to $P_2^M = 108$ in a medium state in period $t = 2$ but fall to $P_2^B = 80$ in a bad state at $t = 2$. Note that since there are two paths along the binominal tree that lead to the medium state at $t = 2$, but only one path that leads to the good state at $t = 2$ and one path that leads to the bad state $t = 2$, the medium state is more likely to occur. In particular, from the perspective of $t = 0$, the medium state at $t = 2$ will occur with probability $1/2$, while the good and bad states at $t = 2$ will each occur with probability $1/4$.

Suppose that, in the meantime, the price of a government bond stays constant at 100 both over time and across states of the world, so that as shown in the binominal tree below, $Q_0 = 100$ is the price of a bond at $t = 0$, $Q_1^G = 100$ and $Q_2^B = 100$ are the prices of a bond in the good and bad states at $t = 1$, and $Q_2^G = 100$, $Q_2^M = 100$, and $Q_2^B = 100$ are the prices of the bond in the good, medium, and bad states at $t = 2$.



The assumption that the bond price remains constant implies that the interest rate on bonds equals zero. We'll make the assumption here, mainly for our own convenience since it will make the calculations easier. However, the assumption of zero interest rates is not too unrealistic these days.

Our goal will be to use this information about the prices of the stock and bond to price a call option that gives the holder the right, but not the obligation, to buy a share of stock at the strike price $K = 100$ at $t = 2$. With reference to the binomial tree for the stock, we can infer that the holder of this option will find it optimal to exercise when it is "in the money" in the good and medium states at $t = 2$ but to allow the option to expire when it is "out of the money" in the bad state at $t = 2$. We can begin constructing the binomial tree for the option itself, therefore, by noting that the option's value will be $V_2^G = 44$ in the good state at $t = 2$, $V_2^M = 8$ in the medium state at $t = 2$, and $V_2^B = 0$ in the bad state at $t = 2$:



As indicated in this same binominal tree, our task that remains is to use no arbitrage arguments to determine the price (or "value") of the option V_1^G and V_1^B in the good and bad states at $t = 1$ and the price of the option V_0 at $t = 0$.

To accomplish these goals, we will work through a process of "backwards recursion," so called because we will start by finding the value of the option in each of the two states at $t = 1$ and then use those results to determine the value of the option at $t = 0$.

a. Start by considering the situation that prevails in the good state at $t = 1$. At that time and in that state, the stock sells for $P_1^G = 120$ and the bond sells for $Q_1^G = 100$. Looking ahead to $t = 2$, the stock price can rise to $P_2^G = 144$ in the good state at $t = 2$ but can fall to $P_2^M = 108$ in the medium state at $t = 2$. In the meantime, the bond's value stays the same, with $Q_2^G = 100$ and $Q_2^M = 100$, no matter what happens between $t = 1$ and $t = 2$. We want to find a portfolio consisting of s shares of stock and b bonds that will replicate the option's payoffs, equal to $V_2^G = 44$ in the good state at $t = 2$ and $V_2^M = 8$ in the medium state at $t = 2$. If we look at the problem in this way, we can see that mathematically, it takes the same form as those we've solved before. To match the option's payoff in the good state, s and b must satisfy

$$144s + 100b = 44$$

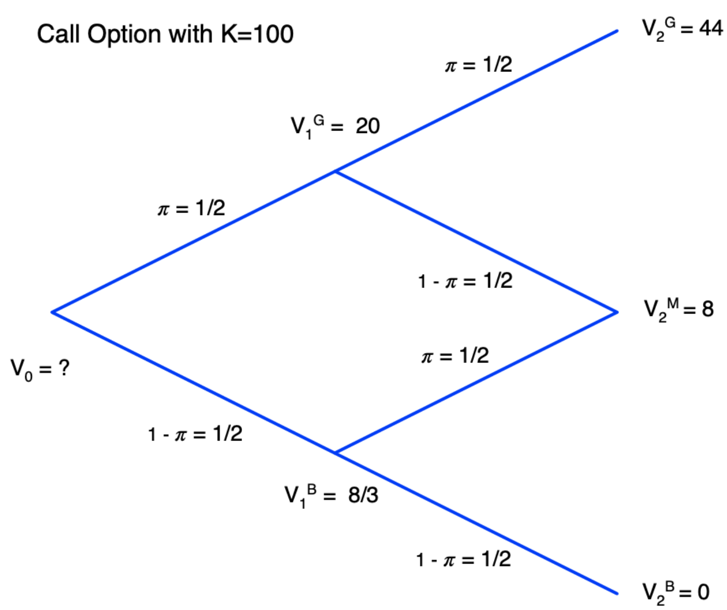
and to match the option's payoff in the medium state, s and b must satisfy

$$108s + 100b = 8.$$

Use this two-equation system to find the numerical values of s and b, the numbers of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from the good state at $t = 1$. Then, use the fact that the stock sells for $P_1^G = 120$ and the bond sells for $Q_1^G = 100$ in the good state at $t = 1$ to compute the price V_1^G of the option in the good state at $t = 1$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

b. Now consider instead the situation that prevails in the bad state at $t = 1$. At that time and in that state, the stock sells for $P_1^B = 90$ and the bond sells for $Q_1^B = 100$. Looking ahead to $t = 2$, the stock price can rise back to $P_2^M = 108$ in the medium state at $t = 2$ but can fall still further to $P_2^B = 81$ in the bad state at $t = 2$. In the meantime, the bond's value stays the same, with $Q_2^M = 100$ and $Q_2^B = 100$, no matter what happens between $t = 1$ and $t = 2$. Once again, we want to find a portfolio consisting of s shares of stock and b bonds that will replicate the option's payoffs, equal to $V_2^M = 8$ in the good state at $t = 2$ and $V_2^B = 0$ in the bad state at $t = 2$. Using all of this information, write down the two equations that s and b must satisfy and use them to find the numerical values of s and b, the number of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from the bad state at $t = 1$. Then, use these values of s and b to compute the price V_1^B of the option in the bad state at $t = 1$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

c. In part (a), you should have found that the call option price is $V_1^G = 20$ in the good state at $t = 1$. And in part (b), you should have found that the call option price is $V_1^B = 8/3 = 2.67$ in the bad state at $t = 1$. Therefore, we can continue to fill in the binomial tree for the option as shown below:



Now, let's step back to $t = 0$, when the stock sells for $P_0 = 100$ and the bond for $Q_0 = 100$. Looking ahead to $t = 1$, we know that the stock price will rise to $P_1^G = 120$ in the good state but fall to $P_1^B = 90$ in the bad state. We also know that the bond price will remain at $Q_1^G = Q_1^B = 100$ no matter what. Once more, we want to find a portfolio consisting of s shares of stock and b bonds to replicate the options payoffs, equal to $V_1^G = 20$ if we move to the good state at $t = 1$ and $V_1^B = 8/3$ if we move to the bad state at $t = 1$. Write down the two equations that s and b must satisfy and use them to find the numerical values of s and b, the number of shares of stock and bonds that must be purchased (if positive) or sold short (if negative) to replicate the option's payoffs looking ahead from $t = 0$ to $t = 1$. Then, use these values of s and b to compute the price V_0 of the option at $t = 0$ assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and options.

2. Using Options to Infer Contingent Claims Prices

In 1978, Douglas Breeden and Robert Litzenberger showed how options on the Standard & Poor's 500 stock index could be used to infer the prices of contingent claims in the real world. Recall from our discussions in class that to do this, they assumed that there are N states of the world, corresponding to different levels of the S&P500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

for some $\delta > 0$. That is, better states of the world correspond to higher levels of the S&P 500, with levels of the S&P 500 arranged on a grid with δ points between each entry.

Next, Breeden and Litzenberger showed that if one constructs a "butterfly" portfolio of call options by buying one call on the S&P 500 with strike price P^{i-1}, writing (selling short) two calls on the S&P 500 with strike price P^i, and buying one call on the S&P 500 with strike price P^{i+1}, then the resulting portfolio will pay off δ dollars in state i, when the S&P 500 is at level $P = P^i$, and zero otherwise. Thus, if q_o^i denotes the price of a call option with strike price P^i, no arbitrage implies that the price q_{cc}^i of a contingent claim that pays off one dollar in state i and zero otherwise can be computed as

$$q_{cc}^i = (1/\delta)(q_o^{i-1} + q_o^{i+1} - 2q_o^i).$$

The table below shows prices (at the close of business on Friday, March 12, when the S&P 500 itself stood at 3943) of call options on the S&P 500 expiring on May 21, 2021, for five strike prices on a grid that sets $\delta = 100$, taken from the "quotes dashboard" on the website of the Chicago Board Options Exchange:

S&P 500 Call Option Prices
May 21, 2021 Expiration

Strike Price	Option Price
$K = P^1 = 3800$	$q_o^1 = 221$
$K = P^2 = 3900$	$q_o^2 = 150$
$K = P^3 = 4000$	$q_o^3 = 91$
$K = P^4 = 4100$	$q_o^4 = 50$
$K = P^5 = 4200$	$q_o^5 = 25$

Use these data, together with Breeden and Litzenberger's formula, to infer the prices of a contingent claim for the states in which the S&P 500 is at $P^2 = 3900$, $P^3 = 4000$ and $P^4 = 4100$ on May 21.