Midterm Exam

ECON 337901 - Financial Economics
Boston College, Department of Economics

Peter Ireland
Spring 2020

Due Thursday, March 26, 12noon

This exam has five questions on six pages, including this cover sheet. Before you begin, please check to make sure that your copy has all five questions and all six pages. Please note, as well, that each of the five questions has three parts. The five questions will be weighted equally in determining your overall exam score.

Please circle your final answer to each part of each question after you write it down, so that I can find it more easily. If you show the steps that led you to your results, however, I can award partial credit for the correct approach even if your final answers are slightly off.

This is an open-book exam, meaning that it is fine for you to consult your notes and all material from the course webpage when working on your answers to the questions. I expect you to work independently on the exam, however, without discussing the questions or answers with anyone else, in person or electronically, inside or outside of the class; the answers you submit must be yours and yours alone.

The exam starts on the next page.

1. Farming

This question asks you to solve a variant of the farming problem that you studied, previously, by working on question 2 of problem set 1. Suppose that, in order to produce c units of output for his or her own consumption, a farmer must work for

$$h = \left(\frac{1}{2A}\right) c^2$$

hours. In this version of the model, $A > 0$ is a positive number that measures the farmer's productivity: when A goes up, the farmer is able to produce the same amount of output with fewer hours worked. Suppose also that, once again, the farmer's preferences over consumption versus hours worked are described by the utility function

$$\ln(c) - \beta h,$$

where ln denotes the natural logarithm and $\beta > 0$ is a positive number that measures the farmer's distaste for work: higher values of β mean that the farmer has a stronger preference for leisure relative to consumption. By substituting the expression for h into the utility function, we can find the farmer's optimal choice of consumption c^* by solving the unconstrained optimization problem

$$\max_c \ \ln(c) - \left(\frac{\beta}{2A}\right) c^2.$$

a. Write down the first-order condition for the farmer's optimal choice of c^*.

b. Next, use your first-order condition to find the solution for c^* in terms of A and β. *Note:* Although you may see that, mathematically, there are two values of c^* that satisfy the first-order condition, the requirement that consumption be positive means that only the solution with $c^* > 0$ makes economic sense.

c. Finally, use your solution to answer the following two questions. What happens to the farmer's optimal consumption c^* when his or her productivity A rises: does c^* increase, decrease, or stay the same? And what happens to the farmer's consumption c^* when his or her aversion to work as measured by β rises: does c^* increase, decrease, or stay the same?

2. Intertemporal Consumer Optimization

Following Irving Fisher, consider a consumer who receives income Y_0 in period $t = 0$ (today), which he or she divides up into an amount c_0 to be consumed and an amount s to be saved (or borrowed, if $s < 0$) subject to the budget constraint

$$Y_0 \geq c_0 + s.$$

Then, in period $t = 1$ (next year), the consumer receives income Y_1, which he or she uses to finance consumption c_1, subject to the budget constraint

$$Y_1 + (1 + r)s \geq c_1,$$

where r denotes the interest rate on saving (and borrowing). As in class, we can combine these two single-period budget constraints into one present-value budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

thereby eliminating s as a separate choice variable.

Suppose, finally, that the consumer's preferences over consumption during the two periods are described by the utility function

$$\ln(c_0) + \beta \ln(c_1),$$

where ln denotes the natural logarithm and β, the discount factor, measures the consumer's patience.

The consumer therefore solves the constrained maximization problem

$$\max_{c_0, c_1} \ \ln(c_0) + \beta \ln(c_1) \text{ subject to } Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

a. As a first step in finding the optimal choices c_0^* and c_1^*, write down the Lagrangian for the consumer's problem. Then, write down the two first-order conditions.

b. Now assume in particular that $Y_0 = 21$, $Y_1 = 0$, $\beta = 3/4$, and $r = 1/3$, so that $1 + r = 4/3$ and $\beta(1 + r) = 1$. Use these values, together with the two first-order conditions you derived in part (a) and the consumer's binding budget constraint

$$Y_0 + \frac{Y_1}{1+r} = c_0^* + \frac{c_1^*}{1+r},$$

to calculate numerical values for the optimal choices c_0^* and c_1^*.

c. Continue to assume that $\beta = 3/4$ and $r = 1/3$, but suppose that instead, $Y_0 = 0$ and $Y_1 = 28$. What are the numerical values for the optimal choice c_0^* and c_1^* in this case?

3. Pricing Contingent Claims

Consider an economic environment in which there are two periods, $t = 0$ (today) and $t = 1$ (next year), and two possible states at $t = 1$: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$.

Suppose, initially, that two assets trade in this economy. A risky stock sells for $q^s = 2.80$ at $t = 0$, $P^G = 6$ in the good state at $t = 1$, and $P^B = 2$ in the bad state at $t = 1$. And a risk-free bond sells for $q^b = 0.90$ at $t = 0$ and pays off 1 in both states at $t = 1$.

As in class, we can find portfolios of the stock and bond that replicate the payoffs on contingent claims, and thereby infer the prices at which those contingent claims should sell for at $t = 0$, if there are to be no arbitrage opportunities across the markets for stocks, bonds, and contingent claims.

a. To begin this process, find the number of shares s and the number of bonds b that an investor would have to buy or sell short in order to replicate the payoff from a contingent claim for the good state. *Note:* For this problem and for the one in part (b), below, one or both of s and b might turn out to be fractions.

b. Next, find the number of shares s and the number of bonds b that an investor would have to buy or sell short in order to replicate the payoff from a contingent claim for the bad state.

c. Finally, use your answers from parts (a) and (b) to compute the time $t = 0$ prices of both contingent claims implied by the payoffs and prices of the stock and bond, assuming that there are no arbitrage opportunities across the markets for stocks, bonds, and contingent claims.

4. Using Stock Options to Manage Risk

Consider another economic environment in which there are two periods, $t = 0$ (today) and $t = 1$ (next year), and two possible states at $t = 1$: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$.

Suppose that, initially, only two assets trade in this economy. The first is a stock, which sells for price $q^s = 2$ at $t = 0$, $P^G = 4$ in the good state at $t = 1$, and $P^B = 2$ in the bad state at $t = 1$. The second is a call option on the stock with strike price $K = 3$, which sells for price $q^o = 0.25$ at $t = 0$.

Following the same logic we discussed in class, an investor who takes a long position in this call option will pay $q^o = 0.25$ for it at $t = 0$ and find it optimal to exercise the option and receive a payoff of $P^G - K = 4 - 3 = 1$ in the good state at $t = 1$. In the bad state at $t = 1$, the call is "out of the money." Therefore, the investor with a long position in the call will allow it to expire, with a payoff of zero, in the bad state. Similar reasoning then shows that an investor who "writes," that is, takes a short position, in the call will receive $q^0 = 0.25$ today and will have to pay 1 in the good state and 0 in the bad state at $t = 1$.

By assumption, no bond is traded in this economy. But can an investor form a portfolio of the two risky assets, the stock and the stock option, to create a "synthetic" risk-free asset? The answer is yes, provided the investor can take short as well as long positions in the two existing assets.

a. To see this, start by letting s be the number of shares of stock and c be the number of options to be included in the desired portfolio. Positive values for s and/or c mean that the investor is taking a long position; negative values indicate that a short position is needed instead. Using the payoffs from the stock and the option, write down two equations that must hold if the portfolio is to match the payoffs on a discount bond: 1 in the good state at $t = 1$ and 1 in the bad state at $t = 1$.

b. Next, solve the two equations you wrote down in part (a) to find the numerical values of s and c that produce the desired, risk-free portfolio (once again, for this problem, one or both of s and c might turn out to be fractions). Then, use your solutions to answer the questions: Does forming the risk-free portfolio involve taking a long or short position in the stock? Does it involve taking a long or short position in the option?

c. Finally, use your solutions to answer the question: what would be the price of the discount bond at $t = 0$ if there are no arbitrage opportunities across the markets for stocks, options, and bonds?

5. Pricing Risk-Free Assets

Suppose that a one-year discount bond that pays off one dollar for sure one year from now sells for $P_1 = 0.90$ today, and a two-year discount bond that pays off one dollar for sure two years from now sells for $P_2 = 0.80$ today.

a. Consider, first, a new risk-free asset, which pays off 100 dollars for sure one year from now and 100 dollars for sure two years from now. What will the price of this asset be if there are no arbitrage opportunities across all markets for risk-free assets?

b. Consider, next, another risk-free asset that pays the holder 100 dollars for sure one year from now but then requires the holder *to pay* 100 dollars for sure two years from now. What will the price of this asset be if there are no arbitrage opportunities across all markets for risk-free assets?

c. Suppose, finally, that yet another risk-free asset, which pays off 100 dollars for sure one year from now, 100 dollars for sure two years from now, and 100 dollars for sure three years from now, is observed to sell for 230 dollars today. Use this observation, together with the data from above on the one and two-year discount bonds, to infer what the price today will be of a three-year discount bond, which pays off one dollar for sure three years from now, if there are no arbitrage opportunities across all markets for risk-free assets.