Solutions to Midterm Exam

ECON 337901 - Financial Economics | Peter Ireland
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Due Tuesday, March 26

1. Utility Maximization

The Lagrangian for the consumer's problem is

$$L(c, l, \lambda) = \ln(c) - \left(\frac{1}{2}\right) l^2 + \lambda[(1-T)wl - c].$$

a. Differentiating the Lagrangian with respect to c and setting the result equal to zero yields

$$\frac{1}{c^*} - \lambda^* = 0.$$

Similarly, differentiating the Lagrangian with respect to l and setting the result equal to zero yields

$$-l^* + \lambda^*(1-T)w = 0.$$

b. To find the solutions for c^* and l^*, start by rewriting the first-order conditions as

$$c^* = \frac{1}{\lambda^*}$$

and

$$l^* = \lambda^*(1-T)w,$$

then substitute these expressions into the binding constraint to obtain

$$\lambda^*(1-T)^2 w^2 = \frac{1}{\lambda^*}$$

or

$$\lambda^{*2} = \frac{1}{(1-T)^2 w^2}.$$

Since λ^*, $1-T$ and w are all positive, this last result requires

$$\lambda^* = \frac{1}{(1-T)w}.$$

Finally, substitute this solution for λ^* back into the first-order conditions to arrive at the solutions

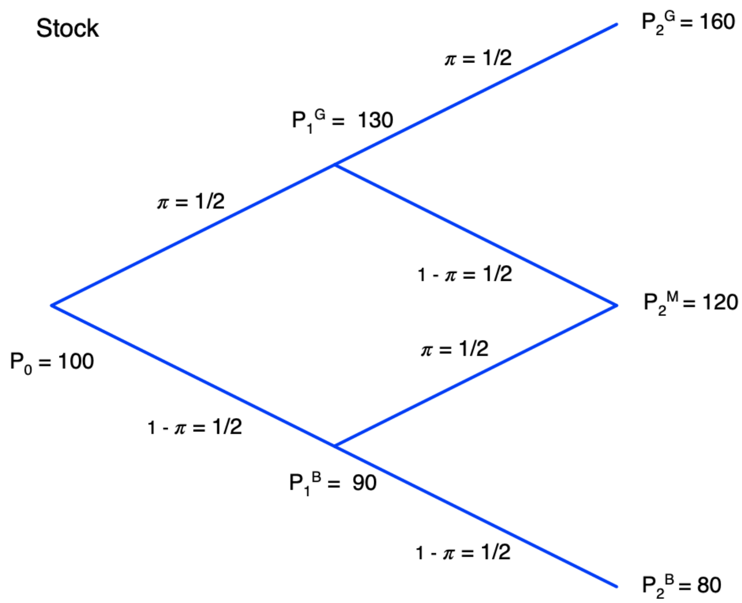
$$c^* = (1-T)w$$

and

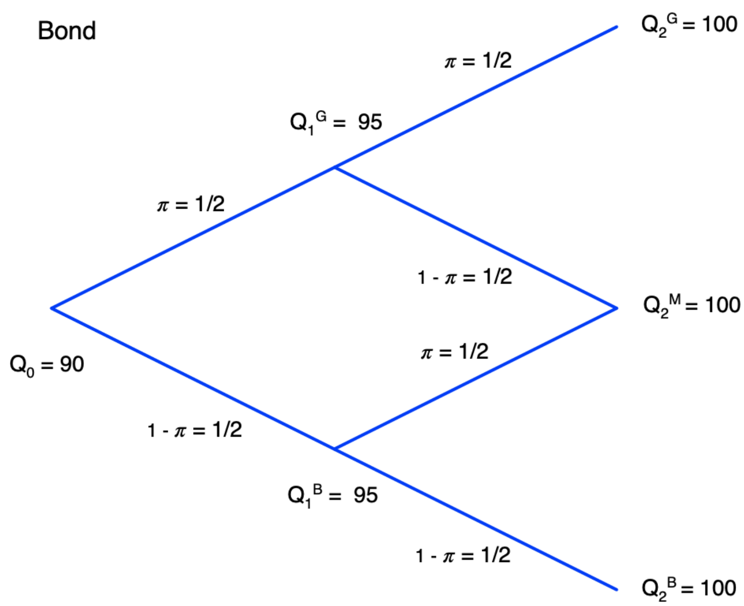
$$l^* = 1.$$

2. Dynamic Hedging

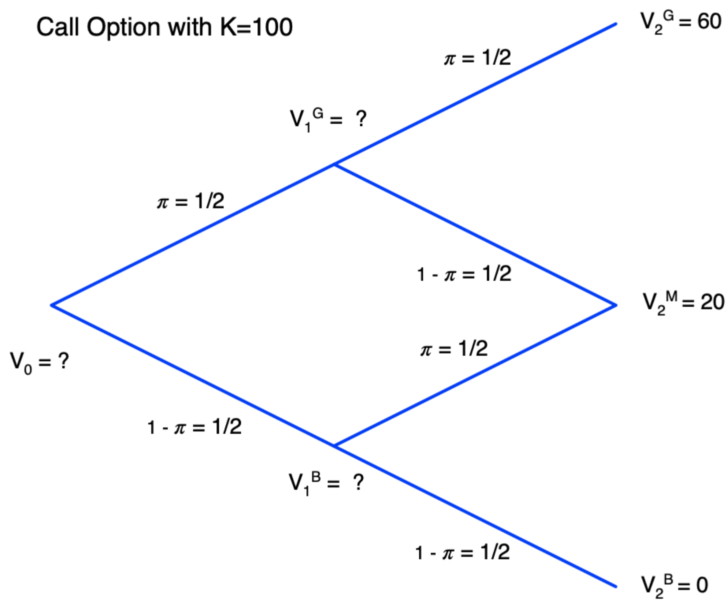
The binomial tree for the price of the stock is



and the binomial tree for the price of the bond is



We want to use the data from these trees to "price" a call option that gives the holder the right, but not the obligation, to buy a share of stock at the strike price $K = 100$ at $t = 2$. The binominal tree for this option is



a. To begin the process of "backwards recursion," start in the good state at $t = 1$. Looking ahead to $t = 2$, the stock price may rise to $P_2^G = 160$ in the good state or fall to $P_2^M = 125$ in the medium state. Either way, the call will be in the money, worth $V_2^G = 60$ in the good state and $V_2^M = 20$ in the medium state. The bond is worth 100 no matter what. Therefore, if s is the number of shares of stock and b the number of bonds required to form the portfolio that replicates the option's payoffs in both states, these values must satisfy

$$160s + 100b = 60$$

and

$$120s + 100b = 20.$$

The easiest way to solve this system of equation is to subtract the second equation from the first to eliminate the term involving b; the result shows that

$$40s = 40$$

or $s = 1$. Plugging this solution for s back into either of the two original equations and solving for b yields $b = -1$. No arbitrage requires the price V_1^G in the good state at $t = 1$ to equal the cost of assembling this portfolio. Since $P_1^G = 130$ is the stock price and $Q_1^G = 95$ is the bond price in the good state at $t = 1$, we now know that

$$V_1^G = 130 - 95 = 35.$$

b. Next, let's move down to the bad state at $t = 1$. Looking ahead to $t = 2$, the stock price may rise to $P_2^M = 120$ in the medium state or fall to $P_2^B = 80$ in the bad state. Now, the call will be in the money, worth $V_2^M = 20$, in the medium state, but out of the money, worth $V_2^B = 0$, in the bad state. The bond is again worth 100 no matter what. Now for the portfolio that replicates the option's payoffs, s and b must satisfy

$$120s + 100b = 20$$

and

$$80s + 100b = 0.$$

Again, it's most convenient to subtract the second equation from the first to eliminate b and solve for

$$40s = 20$$

or $s = 1/2$. Now use either of the two original equations to find $b = -2/5$. No arbitrage requires the price V_1^B in the bad state at $t = 1$ to equal the cost of assembling this portfolio. Since $P_1^B = 90$ is the stock price and $Q_1^B = 95$ is the bond price in the bad state at $t = 1$, we now know that

$$V_1^B = 90 \times (1/2) - 95 \times (2/5) = 7.$$

c. Finally, let's move back to $t = 0$. We've already found that the option will be worth $V_1^G = 35$ in the good state at $t = 1$ and $V_1^B = 7$ in the bad state at $t = 1$. We also know from the original binomial trees that the stock price will be $P_1^G = 130$ in the good state at $t = 1$ and $P_1^B = 90$ in the bad state at $t = 1$. The bond is worth 95 no matter what. Now for the portfolio that replicates the option's payoffs, s and b must satisfy

$$130s + 95b = 35$$

and

$$90s + 95b = 7.$$

Again, it's most convenient to subtract the second equation from the first to eliminate b and solve for

$$40s = 28$$

or $s = 0.7$. Now use either of the two original equations to find $b = -0.5895$. No arbitrage requires the price V_0 at $t = 0$ to equal the cost of assembling this portfolio. Since $P_0 = 100$ is the stock price and $Q_0^B = 90$ is the bond price at $t = 0$, we now know that

$$V_0 = 100 \times 0.7 - 90 \times 0.5895 = 16.95.$$

Before moving on to problem 3, let's view the dynamic hedging strategy required to match the option's payoffs from the perspective of a trader, moving forwards instead of backwards in "real time." At $t = 0$, the solution to part (c) from above shows that this trader must buy $s = 0.7$ shares of stock to replicate the option's payoffs moving from $t = 0$ to $t = 1$.

Now, suppose that the good state arrives at $t = 1$. The solution to part (a) from above shows that, even as the stock price rises from $P_0 = 100$ to $P_1^G = 130$, the trader must increase his or her holdings of the stock to $s = 1$ share. Suppose, on the other hand, that the bad state arrives at $t = 1$. The solution to part (b) from above shows that, even as the stock price falls from $P_0 = 100$ to $P_1^B = 90$, the trader must decrease his or her holdings of the stock to $s = 1/2$ shares.

Thus, a trader using dynamic hedging to track the value of a stock option will have to buy shares when the stock price is rising and sell shares when the stock price is falling. These trading strategies can sometimes work to amplify large stock price movements, as with GameStop in early 2021. Similar strategies have also been blamed for part of the "Black Monday" stock market crash on October 19, 1987, when the S&P 500 stock index declined by more than 20 percent on one day.

3. Futures Pricing

The stock sells for q^s at $t = 0$ and pays off P^G in the good state at $t = 1$, P^M the the medium state at $t = 1$, and P^B in the bad state at $t = 1$. The bond sells for q^b at $t = 0$ and pays off one no matter what at $t = 1$. The futures contract sells for q^F at $t = 0$ and pays off $P^G - F$ in the good state at $t = 1$, $P^M - F$ the the medium state at $t = 1$, and $P^B - F$ in the bad state at $t = 1$.

a. To begin, consider using a portfolio consisting of s shares of stock and b bonds to replicate the payoffs on the futures contract in the good and bad states, ignoring the medium state for now. The values of s and b must satisfy

$$P^G s + b = P^G - F$$

and

$$P^B s + b = P^B - F.$$

Subtract the second equation from the first to get

$$(P^G - P^B)s = P^G - P^B,$$

which implies that $s = 1$. Substitute this solution for s into either equation to get $b = -F$.

b. This portfolio with $s = 1$ and $b = -F$ pays off $P^M - F$ in the medium state at $t = 1$, replicating the futures contract in that third state as well. In fact, when you stop to think about it, this same portfolio of the stock and bond will replicate the payoff on the futures contract no matter how many states of the world there are at $t = 1$. Again, this is something that makes futures pricing a little easier than option pricing.

c. Since the portfolio with $s = 1$ and $b = -F$ costs $q^s - Fq^b$, no arbitrage across all markets implies that the price of the futures contract must be

$$q^F = q^s - Fq^b.$$

d. In practice, the price quoted for futures contracts corresponds to the delivery price F that makes $q^F = 0$. In light of the answer to part (c), this delivery price will be

$$F = q^s/q^b.$$

So long as interest rates, and therefore bond prices, aren't changing, the stock price q^s and the futures price F will move up and down together.

4. Using Options to Infer Contingent Claims Prices

Douglas Breeden and Robert Litzenberger showed how options on the Standard & Poor's 500 stock index could be used to infer the prices of contingent claims in the real world. To do this, they assumed that there are N states of the world, corresponding to different levels of the S&P500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

for some $\delta > 0$. That is, better states of the world correspond to higher levels of the S&P 500, with levels of the S&P 500 arranged on a grid with δ points between each entry.

Next, Breeden and Litzenberger showed that if one constructs a "butterfly" portfolio of call options by buying one call on the S&P 500 with strike price P^{i-1}, writing (selling short) two calls on the S&P 500 with strike price P^i, and buying one call on the S&P 500 with strike price P^{i+1}, then the resulting portfolio will pay off δ dollars in state i, when the S&P 500 is at level $P = P^i$, and zero otherwise. Thus, if q_o^i denotes the price of a call option with strike price P^i, no arbitrage implies that the price q_{cc}^i of a contingent claim that pays off one dollar in state i and zero otherwise can be computed as

$$q_{cc}^i = (1/\delta)(q_o^{i-1} + q_o^{i+1} - 2q_o^i).$$

The table below shows prices during the morning of Wednesday, February 28, 2024 (just before spring break, when the S&P 500 itself stood at 5064) of call options on the S&P 500 expiring on Friday, May 17, 2024 (just before graduation day) for six strike prices on a grid that sets $\delta = 100$, taken from the "quotes dashboard" on the website of the Chicago Board Options Exchange:

S&P 500 Call Option Prices

May 17, 2024 Expiration

Strike Price	Option Price
$K = P^1 = 4800$	$q_o^1 = 344$
$K = P^2 = 4900$	$q_o^2 = 258$
$K = P^3 = 5000$	$q_o^3 = 182$
$K = P^4 = 5100$	$q_o^4 = 118$
$K = P^5 = 5200$	$q_o^5 = 70$
$K = P^6 = 5300$	$q_o^6 = 37$

Besides the genius that lies behind the basic idea, what is truly impressive about Breeden and Litzenberger's results is how easy they are to apply in practice: we can exploit the similarity between option payoffs and contingent claims payoffs to infer contingent claims prices from options prices without having to solve any system of equations!

In particular, to price a contingent claim for the state in which the S&P 500 is at $P^2 = 4900$ on May 17, we only need to plug the relevant options prices into the formula and compute

$$q^2_{cc} = (1/100)(344 + 182 - 2 \times 258) = 0.10.$$

Likewise, for the states in which the S&P 500 is at $P^3 = 5000$, $P^4 = 5100$, and $P^1 = 5200$:

$$q^3_{cc} = (1/100)(258 + 118 - 2 \times 182) = 0.12,$$

$$q^4_{cc} = (1/100)(182 + 70 - 2 \times 118) = 0.16,$$

and

$$q^5_{cc} = (1/100)(118 + 37 - 2 \times 70) = 0.15.$$

This small set of calculations illustrates how we can use option prices to infer contingent claims prices in the real world. As S&P options trade with many strike prices above and below those shown in the table, we can use the same procedure to infer contingent claims prices for even better or worse states of the world. And since S&P 500 options also trade with strike prices at intervals as small as 5 points, we can use the same procedure to price contingent claims for large number of states intermediate to those considered here.