Solutions to Midterm Exam

ECON 337901 - Financial Economics | Peter Ireland
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Due Tuesday, March 29

1. Utility Maximization

The Lagrangian for the consumer's problem is

$$L(c_a, c_b, \lambda) = c_a c_b + \lambda(Y - p_a c_a - p_b c_b).$$

a. Differentiating the Lagrangian with respect to c_a and setting the result equal to zero yields

$$c_b^* - \lambda^* p_a = 0.$$

Similarly, differentiating the Lagrangian with respect to c_b and setting the result equal to zero yields

$$c_a^* - \lambda^* p_b = 0.$$

b. Together with the binding constraint

$$Y = p_a c_a^* + p_b c_b^*,$$

the two first-order conditions from part (a) form a system of three equations in the three unknowns c_a^*, c_b^*, and λ^*. To solve for c_a^* and c_b^*, rewrite the first-order conditions as

$$c_b^* = \lambda^* p_a$$

and

$$c_a^* = \lambda^* p_b$$

and substitute these expressions into the budget constraint to obtain

$$Y = \lambda^* p_a p_b + \lambda^* p_a p_b = 2\lambda^* p_a p_b.$$

Therefore

$$\lambda^* = \frac{Y}{2p_a p_b},$$

$$c_a^* = \frac{Y}{2p_a},$$

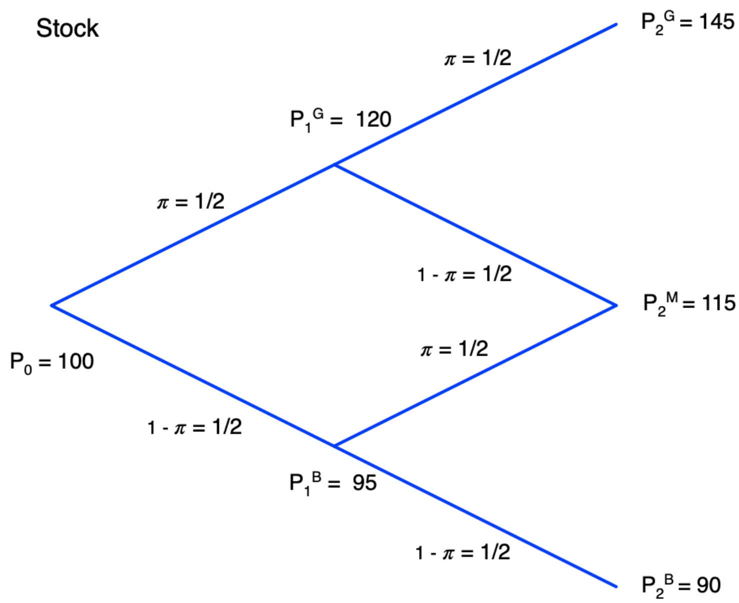
and

$$c_b^* = \frac{Y}{2p_b}.$$

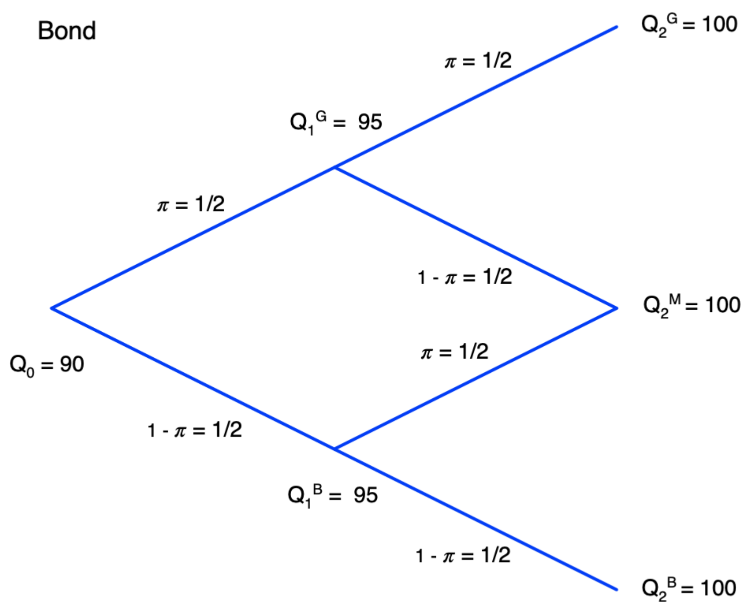
The solutions for c_a^* and c_b^* confirm that in this example, both apples and bananas are normal goods: c_a^* goes up when Y increases and goes down when p_a increases and, similarly, c_b^* goes up when Y increases and goes down when p_b increases.

2. Dynamic Hedging

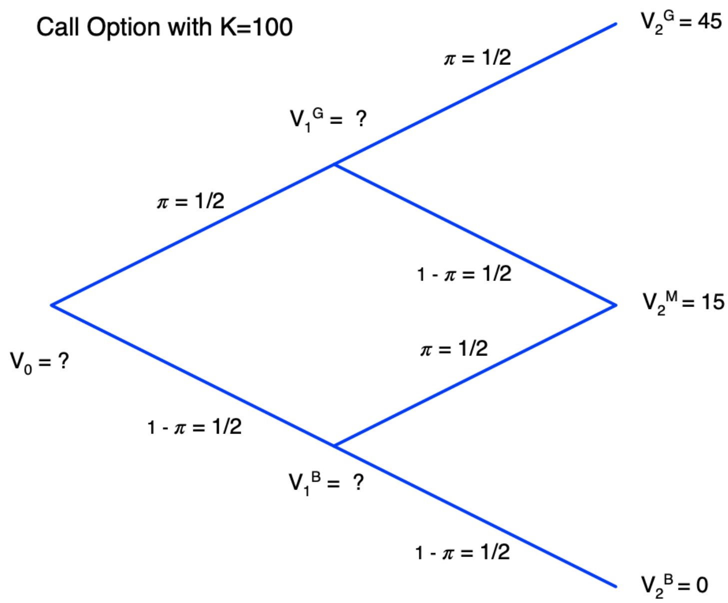
The binomial tree for the price of the stock is



and the binomial tree for the price of the bond is



We want to use the data from these trees to "price" a call option that gives the holder the right, but not the obligation, to buy a share of stock at the strike price $K = 100$ at $t = 2$. The binominal tree for this option is



a. To begin the process of "backwards recursion," start in the good state at $t = 1$. Looking ahead to $t = 2$, the stock price may rise to $P_2^G = 145$ in the good state or fall to $P_2^M = 115$ in the medium state. Either way, the call will be in the money, worth $V_2^G = 45$ in the good state and $V_2^M = 15$ in the medium state. The bond is worth 100 no matter what. Therefore, if s is the number of shares of stock and b the number of bonds required to form the portfolio that replicates the option's payoffs in both states, these values must satisfy

$$145s + 100b = 45$$

and

$$115s + 100b = 15.$$

The easiest way to solve this system of equation is to subtract the second equation from the first to eliminate the term involving b; the result shows that

$$30s = 30$$

or $s = 1$. Plugging this solution for s back into either of the two original equations and solving for b yields $b = -1$. No arbitrage requires the price V_1^G in the good state at $t = 1$ to equal the cost of assembling this portfolio. Since $P_1^G = 120$ is the stock price and $Q_1^G = 95$ is the bond price in the good state at $t = 1$, we now know that

$$V_1^G = 120 - 95 = 25.$$

b. Next, let's move down to the bad state at $t = 1$. Looking ahead to $t = 2$, the stock price may rise to $P_2^M = 115$ in the medium state or fall to $P_2^B = 90$ in the bad state. Now, the call will be in the money, worth $V_2^M = 15$, in the medium state, but out of the money, worth $V_2^B = 0$, in the bad state. The bond is again worth 100 no matter what. Now for the portfolio that replicates the option's payoffs, s and b must satisfy

$$115s + 100b = 15$$

and

$$90s + 100b = 0.$$

Again, it's most convenient to subtract the second equation from the first to eliminate b and solve for

$$25s = 15$$

or $s = 3/5$. Now use either of the two original equations to find $b = -27/50$. No arbitrage requires the price V_1^B in the bad state at $t = 1$ to equal the cost of assembling this portfolio. Since $P_1^B = 95$ is the stock price and $Q_1^B = 95$ is the bond price in the bad state at $t = 1$, we now know that

$$V_1^B = 95(3/5) - 95(27/50) = 5.70.$$

c. Finally, let's move back to $t = 0$. We've already found that the option will be worth $V_1^G = 25$ in the good state at $t = 1$ and $V_1^B = 5.70$ in the bad state at $t = 1$. We also know from the original binomial trees that the stock price will be $P_1^G = 120$ in the good state at $t = 1$ and $P_1^B = 95$ in the bad state at $t = 1$. The bond is worth 95 no matter what. Now for the portfolio that replicates the option's payoffs, s and b must satisfy

$$120s + 95b = 25$$

and

$$95s + 95b = 5.70.$$

Again, it's most convenient to subtract the second equation from the first to eliminate b and solve for

$$25s = 19.30.$$

or $s = 0.7720$. Now use either of the two original equations to find $b = -0.7120$. No arbitrage requires the price V_0 at $t = 0$ to equal the cost of assembling this portfolio. Since $P_0 = 100$ is the stock price and $Q_0^B = 90$ is the bond price at $t = 0$, we now know that

$$V_0 = 100(0.7720) - 90(0.7120) = 13.12.$$

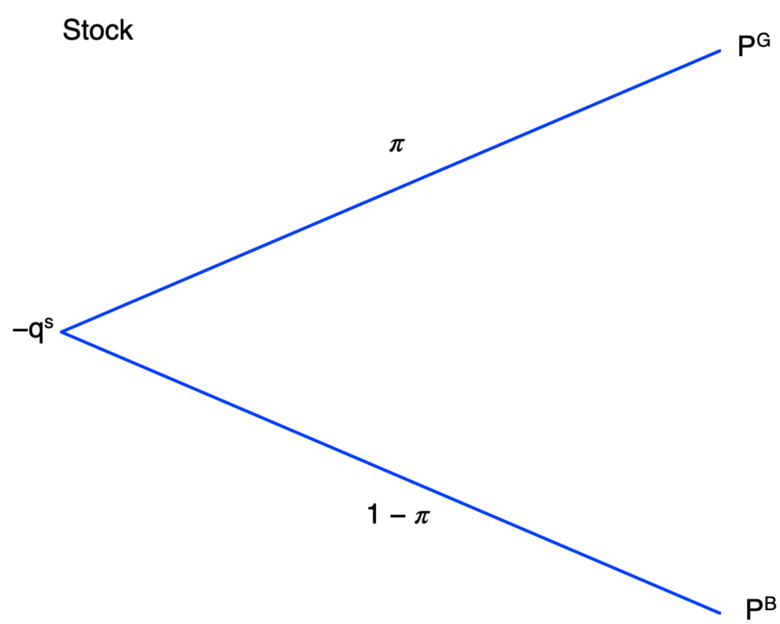
Before moving on to problem 2, let's view the dynamic hedging strategy required to match the option's payoffs from the perspective of a trader, moving forwards instead of backwards in "real time." At $t = 0$, the solution to part (c) from above shows that this trader must buy $s = 0.7720$ shares of stock to replicate the option's payoffs moving from $t = 0$ to $t = 1$.

Now, suppose that the good state arrives at $t = 1$. The solution to part (a) from above shows that, even as the stock price rises from $P_0 = 100$ to $P_1^G = 120$, the trader must increase his or her holdings of the stock to $s = 1$ share. Suppose, on the other hand, that the bad state arrives at $t = 1$. The solution to part (b) from above shows that, even as the stock price falls from $P_0 = 100$ to $P_1^B = 95$, the trader must decrease his or her holdings of the stock to $s = 3/5 = 0.6$ shares.

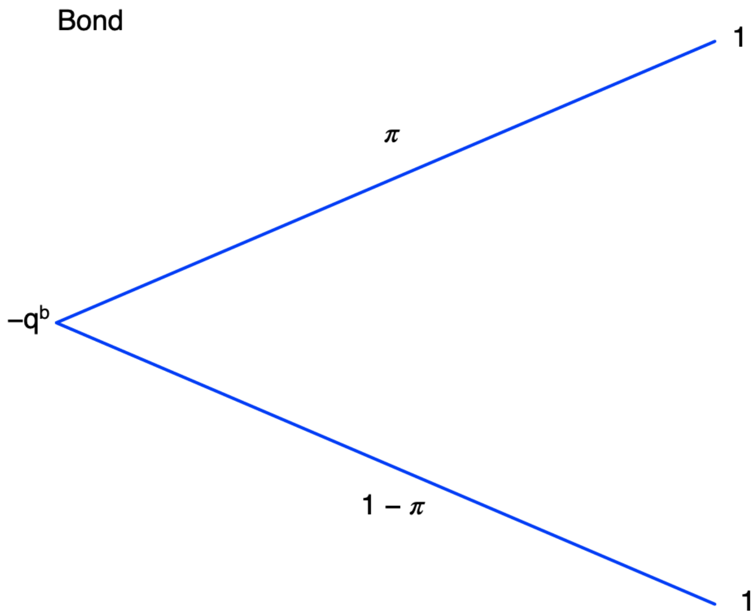
Thus, a trader using dynamic hedging to track the value of a stock option will have to buy shares when the stock price is rising and sell shares when the stock price is falling. These trading strategies can sometimes work to amplify large stock price movements, as with GameStop in early 2021. Similar strategies have also been blamed for part of the "Black Monday" stock market crash on October 19, 1987, when the S&P 500 stock index declined by more than 20 percent on one day.

3. Futures Pricing

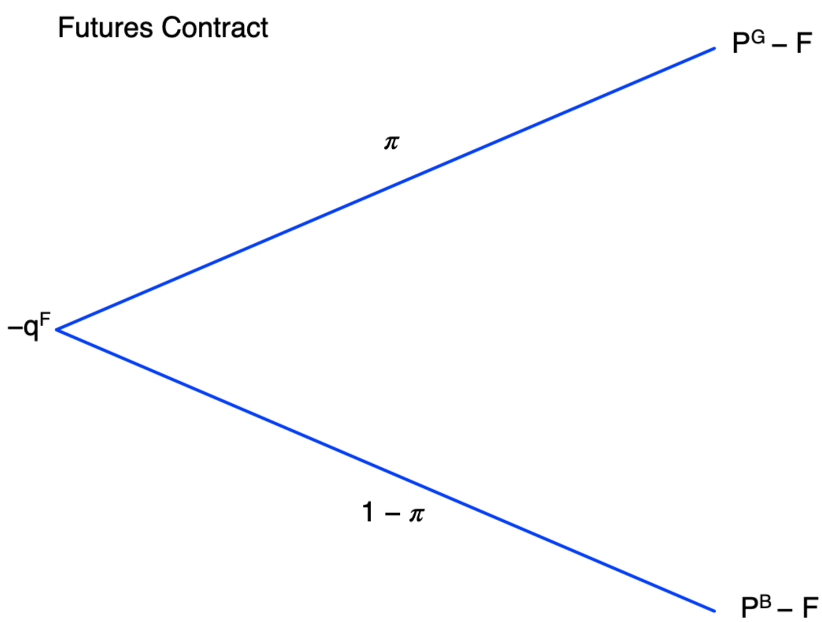
The event tree illustrating cash flows from a long position in the stock is



and the event tree illustrating cash flows from a long position in the bond is



We want to use the data from these trees to "price" a futures contract that gives the buyer the obligation to buy a share of stock at the delivery price F at $t = 1$. The event tree illustrating cash flows from a long position in the futures contract is



a. To begin, we need to find the portfolio consisting of s shares of stock and b bonds that will replicate the payoffs on the futures contract. In the good state, this requires that

$$P^G s + b = P^G - F.$$

And in the bad state, this requires that

$$P^B s + b = P^B - F.$$

b. Probably, the easiest way to solve the two-equation system for s and b is to subtract the second equation from the first to get

$$(P^G - P^B)s = P^G - P^B$$

or $s = 1$. Substituting this solution for s back into either of the two original equations then yields $b = -F$. It might seem surprising that the solutions for s and b are so simple: they don't depend on P^G or P^B at all! But having derived these solutions, we can now look back at the event tree for the futures contract and see why they work. The solution $s = 1$ means, in words, "buy one share of stock." This part of the portfolio pays off P^G in the good state at $t = 1$ and P^B in the bad state at $t = 1$. The solution $b = -F$ means "sell short F bonds." This part of the strategy requires a payment of F in both states at $t = 1$. And so, as required, the portfolio as a whole pays off $P^G - F$ in the good state at $t = 1$ and $P^B - F$ in the bad state at $t = 1$.

c. If there are to be no arbitrage opportunities across the markets for the stock, bond, and futures contract, the price q^F of the futures contract at $t = 0$ must equal the cost of assembling the portfolio with $s = 1$ and $b = -F$ at $t = 0$. Since the stock and bond prices are q^s and q^b at $t = 0$, this means

$$q^F = q^s - q^b F.$$

d. Setting $q^F = 0$ in the solution just derived shows that the quoted "futures price" is

$$F = q^s / q^b.$$

This solution confirms that so long as bond prices and interest rates remain unchanged, F and q^s will move together over the course of a trading day.

4. Using Options to Infer Contingent Claims Prices

Douglas Breeden and Robert Litzenberger showed how options on the Standard & Poor's 500 stock index could be used to infer the prices of contingent claims in the real world. To do this, they assumed that there are N states of the world, corresponding to different levels of the S&P500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

for some $\delta > 0$. That is, better states of the world correspond to higher levels of the S&P 500, with levels of the S&P 500 arranged on a grid with δ points between each entry.

Next, Breeden and Litzenberger showed that if one constructs a "butterfly" portfolio of call options by buying one call on the S&P 500 with strike price P^{i-1}, writing (selling short) two calls on the S&P 500 with strike price P^i, and buying one call on the S&P 500 with strike price P^{i+1}, then the resulting portfolio will pay off δ dollars in state i, when the S&P 500 is at level $P = P^i$, and zero otherwise. Thus, if q_o^i denotes the price of a call option with strike price P^i, no arbitrage implies that the price q_{cc}^i of a contingent claim that pays off one dollar in state i and zero otherwise can be computed as

$$q_{cc}^i = (1/\delta)(q_o^{i-1} + q_o^{i+1} - 2q_o^i).$$

The table below shows prices (during the afternoon of Tuesday, March 8, 2022, when the S&P 500 itself stood at 4250) of call options on the S&P 500 expiring on Friday, May 20, 2022, for six strike prices on a grid that sets $\delta = 100$, taken from the "quotes dashboard" on the website of the Chicago Board Options Exchange:

S&P 500 Call Option Prices

May 20, 2022 Expiration

Strike Price	Option Price
$K = P^1 = 4000$	$q_o^1 = 373$
$K = P^2 = 4100$	$q_o^2 = 301$
$K = P^3 = 4200$	$q_o^3 = 233$
$K = P^4 = 4300$	$q_o^4 = 172$
$K = P^5 = 4400$	$q_o^5 = 119$
$K = P^6 = 4500$	$q_o^6 = 76$

Besides the genius that lies behind the basic idea, what is truly impressive about Breeden and Litzenberger's results is how easy they are to apply in practice: we can exploit the similarity between option payoffs and contingent claims payoffs to infer contingent claims prices from options prices without having to solve any system of equations!

In particular, to price a contingent claim for the state in which the S&P 500 is at $P^2 = 4100$ on April 1, we only need to plug the relevant options prices into the formula and compute

$$q_{cc}^2 = (1/100)(373 + 233 - 2 \times 301) = 0.04.$$

Likewise, for the states in which the S&P 500 is at $P^3 = 4200$, $P^4 = 4300$, and $P^5 = 4400$:

$$q_{cc}^3 = (1/100)(301 + 172 - 2 \times 233) = 0.07,$$

$$q_{cc}^4 = (1/100)(233 + 119 - 2 \times 172) = 0.08,$$

and

$$q_{cc}^5 = (1/100)(172 + 76 - 2 \times 119) = 0.10.$$

This small set of calculations illustrates how we can use option prices to infer contingent claims prices in the real world. As S&P options trade with many strike prices above and below those shown in the table, we can use the same procedure to infer contingent claims prices for even better or worse states of the world. And since S&P 500 options also trade with strike prices at intervals as small as 10 points, we can use the same procedure to price contingent claims for large number of states intermediate to those considered here.