Solutions to Midterm Exam

ECON 337901 - Financial Economics — Peter Ireland
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Due Thursday, March 26, 12noon

1. Farming

The farmer solves the unconstrained optimization problem

$$\max_{c} \ \ln(c) - \left(\frac{\beta}{2A}\right) c^2,$$

where c is his or her consumption and A and β are positive numbers that measure the farmer's productivity and disutility from work.

a. The first-order condition for the farmer's optimal choice of c^* is

$$\frac{1}{c^*} - \left(\frac{\beta}{A}\right) c^* = 0$$

b. Rewrite the first-order condition as

$$\frac{1}{c^*} = \left(\frac{\beta}{A}\right) c^*$$

and rearrange the fractions to get

$$(c^*)^2 = \frac{A}{\beta}.$$

Although, mathematically, there are both positive and negative values of c^* that satisfy this equation, from an economic perspective only the positive solution

$$c^* = \left(\frac{A}{\beta}\right)^{1/2} = \sqrt{\frac{A}{\beta}}$$

makes sense.

c. The solution for c^* shows that the farmer's consumption increases when productivity A goes up, but decreases when his or her aversion to work β goes up.

2. Intertemporal Consumer Optimization

The consumer solves the constrained maximization problem

$$\max_{c_0,c_1} \ \ln(c_0) + \beta \ln(c_1) \text{ subject to } Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

a. The Lagrangian for the consumer's problem is

$$L(c_0, c_1, \lambda) = \ln(c_0) + \beta \ln(c_1) + \lambda \left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right),$$

and two first-order conditions are

$$\frac{1}{c_0^*} - \lambda^* = 0$$

and

$$\frac{\beta}{c_1^*} - \lambda^* \left(\frac{1}{1+r} \right) = 0.$$

b. With $\beta = 3/4$ and $r = 1/3$, the first-order conditions from part (a) imply that

$$c_0^* = \frac{1}{\lambda^*}$$

and

$$c_1^* = \frac{\beta(1+r)}{\lambda^*} = \frac{1}{\lambda^*}.$$

Evidently, it is optimal for consumption to be constant over time. Substituting the values $Y_0 = 21$ and $Y_1 = 0$ into the left-hand side and the common value $1/\lambda^*$ for c_0^* and c_1^* into the right-hand side of the binding constraint

$$Y_0 + \frac{Y_1}{1+r} = c_0^* + \frac{c_1^*}{1+r}$$

yields

$$21 = \frac{1}{\lambda^*} \left(1 + \frac{1}{4/3} \right) = \frac{1}{\lambda^*} \left(1 + \frac{3}{4} \right) = \frac{1}{\lambda^*} \left(\frac{7}{4} \right),$$

implying that

$$\frac{1}{\lambda^*} = \left(\frac{4}{7} \right) 21 = 12$$

and therefore

$$c_0^* = c_1^* = 12.$$

c. With $Y_0 = 0$ and $Y_1 = 28$, the present value of the consumer's income

$$Y_0 + \frac{Y_1}{1+r} = \frac{28}{4/3} = \left(\frac{3}{4} \right) 28 = 21$$

is the same as it was in part (b). Hence, the solutions

$$c_0^* = c_1^* = 12$$

are also the same as before.

3. Pricing Contingent Claims

There are two periods, $t = 0$ and $t = 1$, and two possible states at $t = 1$: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$.

Initially, two assets trade. A risky stock sells for $q^s = 2.80$ at $t = 0$, $P^G = 6$ in the good state at $t = 1$, and $P^B = 2$ in the bad state at $t = 1$. And a risk-free bond sells for $q^b = 0.90$ at $t = 0$ and pays off 1 in both states at $t = 1$.

a. A contingent claim for the good state pays off 1 in the good state in and 0 in the bad. To form a portfolio of s shares and b bonds that replicates these payoffs, s and b must satisfy, in particular,

$$1 = P^G s + b = 6s + b$$

for the good state and

$$0 = P^B s + b = 2s + b$$

for the bad state. Subtracting the second equation from the first reveals that

$$1 = 4s$$

or $s = 1/4$. The second equation then requires

$$b = -2s = -2 \times (1/4) = -1/2.$$

b. A contingent claim for the good state pays off 0 in the good state in and 1 in the bad. To form a portfolio of s shares and b bonds that replicates these payoffs, s and b must satisfy, in particular,

$$0 = P^G s + b = 6s + b$$

for the good state and

$$1 = P^B s + b = 2s + b$$

for the bad state. Subtracting the second equation from the first reveals that

$$-1 = 4s$$

or $s = -1/4$. The first equation then requires

$$b = -6s = 3/2.$$

c. Since the payoffs from the contingent claim for the good state can be replicated by a portfolio that sets $s = 1/4$ and $b = -1/2$, no arbitrage requires that the price of the contingent claim equal the cost of assembling the portfolio:

$$q^G = q^s s + q^b b = 2.80 \times (1/4) + 0.90 \times (-1/2) = 0.70 - 0.45 = 0.25.$$

And since the payoffs from the contingent claim for the bad state can be replicated by a portfolio that sets $s = -1/4$ and $b = 3/2$, no arbitrage requires that the price of the contingent claim equal the cost of assembling the portfolio:

$$q^B = q^s s + q^b b = 2.80 \times (-1/4) + 0.90 \times (3/2) = -0.70 + 1.35 = 0.65.$$

4. Using Stock Options to Manage Risk

There are again two periods, $t = 0$ and $t = 1$, and two possible states at $t = 1$: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$.

A stock sells for price $q^s = 2$ at $t = 0$, $P^G = 4$ in the good state at $t = 1$, and $P^B = 2$ in the bad state at $t = 1$. A call option on the stock with strike price $K = 3$, sells for price $q^o = 0.25$ at $t = 0$.

The investor wants to combine these two assets into a risk-free portfolio that replicates the payoffs on a discount bond.

a. In the good state, the discount bond has a payoff of 1, the stock has a payoff of 4, and the option has a payoff of 1. Therefore, the number of shares s and options c in the portfolio must satisfy

$$1 = 4s + c.$$

In the bad state, the discount bond has a payoff of 1, the stock has a payoff of 2, and the option has a payoff of 0. Therefore, s and c must also satisfy

$$1 = 2s.$$

b. The second equation derived above requires that $s = 1/2$. The first equation then implies

$$c = 1 - 4s = 1 - 4 \times (1/2) = -1.$$

Evidently, the portfolio consists of a long position in 1/2 share of stock and a short position in 1 option.

c. If there are no arbitrage opportunities across the markets for stocks, options, and bonds, the price of the bond at $t = 0$ must equal the cost of assembling the portfolio of the stock and the option. Therefore, the bond price will be

$$q^b = q^s s + q^o c = 2 \times (1/2) + 0.25 \times (-1) = 1 - 0.25 = 0.75.$$

5. Pricing Risk-Free Assets

A one-year discount bond that pays off one dollar for sure one year from now sells for $P_1 = 0.90$ today, and a two-year discount bond that pays off one dollar for sure two years from now sells for $P_2 = 0.80$ today.

a. A new risk-free asset pays off 100 dollars for sure one year from now and 100 dollars for sure two years from now. These payoffs can be replicated by buying a portfolio of 100 one-year discount bonds and 100 two-year discount bonds. Therefore, if there are no arbitrage opportunities across all markets for risk-free assets, the price of this new asset will equal the cost of assembling the portfolio:

$$100P_1 + 100P_2 = 90 + 80 = 170.$$

b. Another new risk-free asset that pays the holder 100 dollars for sure one year from now but then requires the holder *to pay* 100 dollars for sure two years from now. These payoffs can be replicated by constructing a portfolio consisting of a long position in 100 one-year discount bonds and a short position in 100 two-year discount bonds. If there are no arbitrage opportunities across all markets for risk-free assets, the price of this second new risk-free asset will equal the cost of assembling the portfolio:

$$100P_1 - 100P_2 = 90 - 80 = 10.$$

c. Yet another risk-free asset pays off 100 dollars for sure one year from now, 100 dollars for sure two years from now, and 100 dollars for sure three years from now, and is observed to sell for 230 dollars today. These payoffs can be replicated by buying a portfolio of 100 one-year discount bonds, 100 two-year discount bonds, and 100 three-year discount bonds. If there are no arbitrage opportunities across all markets for risk-free assets, the price of the third new risk-free asset will equal the cost of assembling this portfolio:

$$230 = 100P_1 + 100P_2 + 100P_3,$$

where P_3 is the price of a three-year discount bond. Since $P_1 = 0.90$ and $P_2 = 0.80$, this no-arbitrage condition implies that

$$P_3 = \frac{230 - 90 - 80}{100} = \frac{60}{100} = 0.60.$$