Solutions to Midterm Exam

ECON 337901 - Financial Economics | Peter Ireland
Boston College, Department of Economics | Spring 2019

Tuesday, March 19, 10:30 - 11:45am

1. Profit Maximization

With the production function $y = n^{1/2}$, where y is output and n is the number of workers, and with w denoting the real wage, the profit-maximizing firm solves

$$\max_n n^{1/2} - wn.$$

a. The first-order condition for n^* is

$$(1/2)(n^*)^{-1/2} - w = 0.$$

b. To see how the firm's optimal choice n^* depends on w, start by rewriting the first-order condition as

$$(1/2)(n^*)^{-1/2} = w.$$

Next, multiply both sides of the equation by $(n^*)^{1/2}$ and divide both sides by w to get

$$(n^*)^{1/2} = \frac{1}{2w}.$$

Finally, square both sides to get the desired solution

$$n^* = \left(\frac{1}{2w}\right)^2 = \frac{1}{4w^2}.$$

c. The solution from part (b) shows that when w goes up, n^* falls. This makes sense: the firm's demand for labor is a downward-sloping function of the wage w.

2. Consumer Optimization Under Uncertainty

The consumer chooses c_0, c_1^G, and c_1^B to maximize expected utility

$$\ln(c_0) + \left(\frac{1}{4}\right)\ln(c_1^G) + \left(\frac{1}{4}\right)\ln(c_1^B)$$

subject to the budget constraint

$$90 \geq c_0 + \left(\frac{1}{4}\right)c_1^G + \left(\frac{1}{4}\right)c_1^B.$$

a. The Lagrangian for the consumer's problem is

$$L(c_0, c_1^G, c_2^B, \lambda) = \ln(c_0) + \left(\frac{1}{4}\right)\ln(c_1^G) + \left(\frac{1}{4}\right)\ln(c_1^B) + \lambda\left[90 - c_0 - \left(\frac{1}{4}\right)c_1^G - \left(\frac{1}{4}\right)c_1^B\right].$$

b. The first-order conditions for the consumer's optimal choices c_0^*, c_1^{G*}, and c_1^{B*} of c_0, c_1^G, and c_1^B are

$$\frac{1}{c_0^*} - \lambda^* = 0,$$

$$\left(\frac{1}{4}\right)\frac{1}{c_1^{G*}} - \lambda^*\left(\frac{1}{4}\right) = 0,$$

and

$$\left(\frac{1}{4}\right)\frac{1}{c_1^{B*}} - \lambda^*\left(\frac{1}{4}\right) = 0.$$

c. The first-order conditions from part (b) above, imply that

$$c_0^* = c_1^{G*} = c_1^{B*} = \frac{1}{\lambda^*}.$$

Substituting these conditions into the binding budget constraint yields

$$90 = c_0^* + \left(\frac{1}{4}\right)c_1^{G*} + \left(\frac{1}{4}\right)c_1^{B^*} = \frac{1}{\lambda^*}\left(1 + \frac{1}{4} + \frac{1}{4}\right) = \frac{1}{\lambda^*}\left(\frac{6}{4}\right)$$

or

$$\frac{1}{\lambda^*} = 90\left(\frac{4}{6}\right) = \frac{360}{6} = 60,$$

Therefore, the consumer's optimal choices are

$$c_0^* = c_1^{G*} = c_1^{B*} = 60.$$

3. Stocks, Bonds, and Contingent Claims

Two assets are initially traded. A stock sells for $q^s = 2$ today and pays off a large dividend $d^G = 4$ in a good state next year and a smaller dividend $d^B = 2$ in a bad state next year. A bond sells for $q^b = 0.75$ today and pays off one dollar for sure, in both the good and the bad states, next year.

a. To find the price of a contingent claim for the good state, consider forming a portfolio consisting of s shares of stock and b bonds that replicates the claim's payoffs at $t = 1$. In the good state, the claim pays off 1 and the portfolio pays off $4s + b$. Therefore,

$$1 = 4s + b.$$

In the bad state, the claim pays off 0 and the portfolio pays off $2s + b$. Therefore

$$0 = 2s + b.$$

Subtract the second of these equations from the first to get

$$1 = 2s$$

or $s = 1/2$. Substitute this solution for s back into the second equation to get

$$b = -2s$$

or $b = -1$. No arbitrage implies that the price of the claim must equal to the cost of assembling the portfolio. Therefore,

$$q^G = q^s s + q^b b = (1/2)2 + 0.75(-1) = 0.25.$$

b. Similarly, to find the price of a contingent claim for the bad state, consider forming a portfolio consisting of s shares of stock and b bonds that replicates the claim's payoffs at $t = 1$. In the good state, the claim pays off 0 and the portfolio pays off $4s + b$. Therefore,

$$0 = 4s + b.$$

In the bad state, the claim pays off 1 and the portfolio pays off $2s + b$. Therefore

$$1 = 2s + b.$$

Rewrite the first equation as

$$b = -4s$$

and substitute into the second to get

$$1 = 2s + b = 2s - 4s = -2s$$

or $s = -1/2$. Substitute this solution for s back into the first equation to get $b = 2$. No arbitrage implies that the price of the claim must equal to the cost of assembling the portfolio. Therefore,

$$q^B = q^s s + q^b b = (-1/2)2 + 0.75(2) = 0.50.$$

c. The new asset pays off $X^G = 3$ in the good state and $X^B = 1$ in the bad state next year (at $t = 1$). This asset's payoffs next year can be replicated by a portfolio consisting of 3 claims for the good state and 1 claim for the bad state. No arbitrage implies that the price of the new asset must equal the cost of assembling the portfolio of contingent claims. Therefore,

$$q^A = 3q^G + q^B = 3(0.25) + 0.50 = 1.25.$$

Notice that the new asset's payoffs next year can also be replicated by a portfolio formed by taking a long position in one share of stock ($s = 1$) and a short position in one bond ($b = -1$). No arbitrage also implies that the price of the new asset must equal the cost of assembling this portfolio of the stock and the bond. Therefore,

$$q^A = q^s s + q^b b = 2 + 0.75(-1) = 1.25.$$

Either way, the answer is the same.

4. Option Pricing

Here, a stock sells for $q^s = 2$ today (at $t = 0$), and sells for a high price $P^G = 5$ in a good state next year ($t = 1$) and a smaller price $P^B = 3$ in a bad state next year. A bond sells for $q^b = 0.60$ today and pays off one dollar for sure, in both the good and the bad states next year.

a. A call option with $K = 2$ will always be "in the money" and should therefore always be exercised at $t = 1$. It follows that, in the good state, the option's payoff is

$$C^G = P^G - K = 5 - 2 = 3$$

and, in the bad state, the option's payoff is

$$C^B = P^B - K = 3 - 2 = 1.$$

b. Consider a portfolio consisting of s shares of stock and b bonds that replicates the payoffs on the option. In the good state, the call option pays off 3, and the portfolio pays off $5s + b$. Therefore,

$$3 = 5s + b.$$

In the bad state, the call option pays off 1, and the portfolio pays off $3s+b$. Therefore,

$$1 = 3s + b.$$

Subtracting the second equation from the first yields

$$2 = 2s$$

or $s = 1$. Substituting this solution for s into the second equation yields

$$1 = 3s + b = 3 + b$$

or $b = -2$.

c. No arbitrage implies that the price of the option must equal to cost of assembling this portfolio of the stock and the bond. Therefore,

$$q^o = q^s s + q^b b = 2 + 0.60(-2) = 0.80.$$

5. Pricing Risk-Free Assets

A one-year discount bond sells for $P_1 = 0.90$, a two-year discount bond sells for $P_2 = 0.80$ today, and a three-year discount bond sells for $P_3 = 0.70$ today.

a. A three-year coupon bond makes an interest payment of 100 dollars at the end of each of the next three years and also returns face value of 1000 dollars at the end of the third year. These cash flows can be replicated by buying 100 one-year discount bonds, 100 two-year discount bonds, and 1100 three-year discount bonds. No arbitrage implies that the price of the coupon bond must equal the cost of assembling the portfolio of discount bonds. Therefore, the coupon bond's price will be

$$P_3^C = 100(0.90) + 100(0.80) + 1100(0.70) = 90 + 80 + 770 = 940.$$

b. A risk free asset makes payments of 100 dollars two years from now and 100 dollars three years from now. These cash flows can be replicated by buying 100 two-year discount bonds and 100 three-year discount bonds. No arbitrage implies that the price of the asset must equal the cost of assembling the portfolio of discount bonds. Therefore, the asset's price will be

$$100(0.80) + 100(0.70) = 80 + 70 = 150.$$

c. Another risk-free asset pays the holder 100 for sure two years from now but then requires the holder *to pay* 100 for sure three years from now. These cash flows can be replicated by buying 100 two-year discount bonds and selling short 100 three-year discount bonds. No arbitrage implies that the price of the asset must equal the cost of assembling the portfolio of discount bonds. Therefore, the asset's price will be

$$100(0.80) - 100(0.70) = 80 - 70 = 10.$$