Solutions to Final Exam

ECON 337901 - Financial Economics | Peter Ireland
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1. Criteria for Choice Over Risky Prospects

The table below shows the percentage returns on two risky assets, asset 1 and asset 2, in an economic environment in which there are two future states: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$.

	Percentage Return in				
Asset	Good State	Bad State	$E(\tilde{R})$	$\sigma(\tilde{R})$	$E(\tilde{R})/\sigma(\tilde{R})$
Asset 1	15	5	10	5	2
Asset 2	12	6	9	3	3

The table also shows the expected return $E(\tilde{R})$, the standard deviation of the random return $\sigma(\tilde{R})$, and the Sharpe ratio $E(\tilde{R})/\sigma(\tilde{R})$ of each asset, computed as follows:

$$E(\tilde{R}_1) = (1/2)15 + (1/2)5 = 10,$$

$$\sigma(\tilde{R}_1) = [(1/2)(15-10)^2 + (1/2)(5-10)^2]^{1/2} = 5,$$
$$E(\tilde{R}_1)/\sigma(\tilde{R}_1) = 10/5 = 2$$
$$E(\tilde{R}_2) = (1/2)12 + (1/2)6 = 9,$$
$$\sigma(\tilde{R}_2) = [(1/2)(12-9)^2 + (1/2)(6-9)^2]^{1/2} = 3,$$
$$E(\tilde{R}_2)/\sigma(\tilde{R}_2) = 9/3 = 3.$$

As we discussed in class, the formula $E(\tilde{R})/\sigma(\tilde{R})$ for the Sharpe ratio reflects our simplifying assumption that the risk-free interest rate is zero. From the various entries in the table, we can now answer each of the questions below.

a. Does either asset display state-by-state dominance over the other? No! Asset 1 has a higher return in the good state, but asset 2 has a higher return in the bad state.

b. Does either asset display mean-variance dominance over the other? No! Asset 1 has a higher expected return, but also a higher standard deviation of its random return, compared to asset 2.

c. Does either asset have a Sharpe ratio that is larger than the other? Yes! Asset 2 has the higher Sharpe ratio.

2. Insurance

A consumer with initial income of \$100,000 faces a 1 percent probability of experiencing a loss of 90 percent of his or her income. This investor has preferences described by a von Neumann-Morgenstern expected utility function with Bernoulli utility function of the natural log form: $u(Y) = \ln(Y)$.

a. If the consumer does not buy insurance, he or she will have income of \$100,000 with 99 percent probability and income \$10,000 with 1 percent probability. His or her expected utility is therefore
$$0.99 \ln(100000) + 0.01 \ln(10000).$$

b. If the consumer buys insurance, he or she will have income of $100000 - x$ no matter what, where x is the cost of the insurance. Expected utility is therefore
$$\ln(100000 - x).$$

c. The largest amount x^* that the consumer will pay for insurance is the value of x that makes him or her indifferent between buying and not buying insurance. This value x^* satisfies
$$\ln(100000 - x^*) = 0.99 \ln(100000) + 0.01 \ln(10000)$$
or
$$\exp[\ln(100000 - x^*)] = \exp[0.99 \ln(100000) + 0.01 \ln(10000)]$$
$$100000 - x^* = \exp[0.99 \ln(100000) + 0.01 \ln(10000)]$$
$$x^* = 100000 - \exp[0.99 \ln(100000) + 0.01 \ln(10000)] = 100000 - 97724 = 2276.28.$$
Evidently, the consumer will pay up to \$2,276.28 to insure against the loss.

3. Wealth, Risk Aversion, and Portfolio Allocation

An investor with initial wealth $Y_0 = 1000$ allocates the amount a to stocks, which provide return $r_G = 0.25$ in a good state next year that occurs with probability $\pi = 1/2$ and return $r_B = -0.20$ (a 20 percent loss) in a bad state next year that occurs with probability $1 - \pi = 1/2$. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides a return $r_f = 0$ in both states next year. The investor's preferences can be described by a von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the constant relative risk aversion form
$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma},$$
where, in particular, $\gamma = 1/2$.

Therefore, the investor solves
$$\max_a \; \frac{1}{2}\left[\frac{(1000 + 0.25a)^{1/2} - 1}{1/2}\right] + \frac{1}{2}\left[\frac{(1000 - 0.20a)^{1/2} - 1}{1/2}\right].$$

a. The first-order condition for the investor's optimal choice of a^* is

$$\frac{1}{2}\left[\frac{0.25}{(1000+0.25a^*)^{1/2}}\right]-\frac{1}{2}\left[\frac{0.20}{(1000-0.20a^*)^{1/2}}\right]=0.$$

This first-order condition implies that

$$\frac{1}{2}\left[\frac{0.25}{(1000+0.25a^*)^{1/2}}\right]=\frac{1}{2}\left[\frac{0.20}{(1000-0.20a^*)^{1/2}}\right]$$

$$\left(\frac{1000-0.20a^*}{1000+0.25a^*}\right)^{1/2}=\frac{0.20}{0.25}$$

$$\left(\frac{1000-0.20a^*}{1000+0.25a^*}\right)^{1/2}=\frac{4}{5}$$

$$\frac{1000-0.20a^*}{1000+0.25a^*}=\left(\frac{4}{5}\right)^2$$

$$\frac{1000-0.20a^*}{1000+0.25a^*}=\frac{16}{25}.$$

$$25000-5a^*=16000+4a^*.$$

$$9000=9a^*$$

$$a^*=1000.$$

Evidently, in this case, it is optimal for the investor to allocate all of his or her wealth to stocks.

b. We know from our in-class discussions that an investor with constant relative risk aversion will choose an optimal share of initial wealth a^*/Y_0 to allocate to the stock market, and will therefore increase or decrease a^* proportionally as Y_0 changes. Therefore, even without solving the problem again, we know that if initial wealth doubles to $Y_0 = 2000$, the amount allocated to stocks will double as well, so that $a^* = 2000$.

c. We also know from our in-class discussions that more risk averse investors will allocate less to the stock market. Although we would have to solve the problem again to find the exact value of a^* if the investor's constant coefficient of relative risk aversion rises to 2, we can say for sure without re-solving the problem that the value of a^* will be smaller when $\gamma = 2$ than it is in part (a), where $\gamma = 1/2$.

4. A Risk-Return Tradeoff

An investor allocates the share w of his or her initial wealth to a stock mutual fund with risky (random) return $\tilde{r}$, expected return μ_r, and standard deviation of its random return σ_r, and the remaining share $1-w$ to a risk-free bond with return r_f. Stocks are risky, since $\sigma_r > 0$, but have a higher expected return than bonds, since $\mu_r > r_f$.

a. In general, the random return $\tilde{r}_p$ on the investor's overall portfolio will be a weighted average of the return on its two component, in this case the stock mutual fund and the risk-free bond, with weights equal to the portfolio shares. Therefore

$$\tilde{r}_p = w\tilde{r} + (1-w)r_f.$$

From this relationship, it follows that the expected return on the overall portfolio is also a weighted average of the expected return on stocks and the risk free rate:

$$\mu_p = E(\tilde{r}_p) = E[w\tilde{r} + (1-w)r_f] = wE(\tilde{r}) + (1-w)r_f = w\mu_r + (1-w)r_f.$$

By rewriting this last expression as

$$\mu_p = r_f + w(\mu_r - r_f)$$

and recalling the assumption that $\mu_r > r_f$, we can see that the expected return μ_p on the overall portfolio will rise if the investor chooses a larger value of w.

b. In general, it is not true that the standard deviation of a portfolio's random return is a weighted average of the standard deviations of the returns on the individual assets contained in that portfolio. In this special case, however, where there is only one risky asset and one safe asset, the formulas for the portfolio's return and expected return imply that the variance of its random return is

$$\begin{aligned}
\sigma_p^2 &= E[(\tilde{r} - \mu_r)^2] \\
&= E\{[w\tilde{r} + (1-w)r_f - w\mu_r - (1-w)r_f]^2\} \\
&= E\{[w(\tilde{r} - \mu_r)]^2\} \\
&= w^2 E[(\tilde{r} - \mu_r)^2] \\
&= w^2 \sigma_r^2.
\end{aligned}$$

Under the assumption that $w \geq 0$, therefore, the standard deviation of its random return is

$$\sigma_p = w\sigma_r.$$

This expression shows that the standard deviation of the return on the investor's portfolio will also rise if the investor chooses a larger value of w.

c. To derive the formula that relates μ_p directly to σ_p and thereby summarizes the risk-return tradeoff faced by the investor, rewrite the formula for the portfolio's standard deviation as

$$w = \frac{\sigma_p}{\sigma_r}$$

and substitute this expression for w into the formula for the portfolio's expected return:

$$\mu_p = r_f + w(\mu_r - r_f) = r_f + \left(\frac{\sigma_p}{\sigma_r}\right)(\mu_r - r_f) = r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_p$$

This expression shows that the relationship between μ_p and σ_p is linear with y-intercept equal to r_f and slope equal to the risky mutual fund's Sharpe ratio.

5. Using the CAPM to Price a Risky Cashflow

The risk-free interest rate on one-year government bonds is $r_f = 0.01$ and the risky return $\tilde{r}_M$ over the next year on the stock market as a whole has expected value $E(\tilde{r}_M) = 0.07$ and variance $\sigma_M^2 = 0.03$.

Asset A sells for price P^A today and makes a random payoff $\tilde{C}$ one year from now, with $E(\tilde{C^A}) = 105$. Asset A's random return $\tilde{r}_A$ is assumed to be normally distributed, with variance $\sigma_A^2 = 0.04$ and covariance $\text{cov}(\tilde{r}_A, \tilde{r}_M) = \sigma_{AM} = 0.02$ with the market's return.

a. Asset A's capital asset pricing model (CAPM) beta is

$$\beta_A = \frac{\sigma_{AM}}{\sigma_M^2} = \frac{0.02}{0.03} = \frac{2}{3}.$$

b. According to the CAPM, asset A's expected return should be

$$E(\tilde{r}_A) = r_f + \beta_A[E(\tilde{r}_M) - r_f] = 0.01 + \frac{2}{3} \times (0.07 - 0.01) = 0.01 + 0.04 = 0.05,$$

or 5 percent. Since asset A's expected return is related to its price via

$$E(\tilde{r}_A) = \frac{E(\tilde{C^A})}{P^A} - 1$$

and since its expected payoff is given as $E(\tilde{C^A}) = 105$, the CAPM also implies that

$$0.05 = \frac{105}{P^A} - 1$$

$$1.05 = \frac{105}{P^A}.$$

$$P^A = \frac{105}{1.05} = 100.$$

c. If, instead, asset A's random return has covariance $\text{cov}(\tilde{r}_A, \tilde{r}_M) = \sigma_{AM} = 0.03$ with the market's return, its CAPM beta will be higher, its expected return will be higher, and its price today will be lower, than the values computed above. Intuitively, the higher covariance with the market return means that investors will find asset A less useful for diversification; it will therefore have to sell at a lower price.